1 Normed spaces

Definition (Norm over a vector space)**.** Let x be an $\mathbb{R}$ or $\mathbb{C}$-vector space. A map $\|\cdot\| : X \to \mathbb{R}$ is called a **norm** over X if

1. $\forall \alpha \in F : \forall x \in X : \|\alpha x\| = |\alpha| \|x\|$
2. $\forall x, y \in X : \|x + y\| \leq \|x\| + \|y\|$
3. $\forall x \in X : \|x\| = 0 \iff x = 0_X$

The ordered pair $(X, \|\cdot\|)$ is called a **normed** (vector) **space**.

Example. Recall the following norms on $C(I, \mathbb{R})$, where I is a compact set on $\mathbb{R}$, so for example, $a < b$ real, and $I = [a, b]$. We have the definition

$$C(I, \mathbb{R}) := \{f : I \to \mathbb{R} : f \text{ continuous}\}$$

equipped with pointwise addition and scaling, making $C(I, \mathbb{R})$ into an $\mathbb{R}$-vector space. We then define the norms for some $f \in C(I, \mathbb{R})$:

- $\|f\|_1 := \int_a^b |f(t)| \, dt$
- $\|f\|_2 := \left(\int_a^b (f(t))^2 \, dt \right)^{\frac{1}{2}}$
- $\|f\|_\infty := \sup_{t \in I} f(t)$

Definition. Let $(X, \|\cdot\|)$ be a normed space. We induce the metric $d : X \times X \to R$ as follows

$$d(x, y) := \|x - y\|$$

Furthermore, from Analysis we already know how convergence works with respect to this metric by analyzing the metric space (X, d).

Definition. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed vector spaces. We say that a map $f : X \to Y$ is continuous if for all $x \in X$ we have that

$$\lim_{t \to x} f(t) = f(x)$$

Equivalently,

$$\|f(t) - f(x)\| \xrightarrow{t \to x} 0$$

Lemma. $\|\cdot\|$ *is continuous (with respect to standard norm on* $\mathbb{R}$*).*

Proof. Let $x \in X$, $\varepsilon > 0$. Let $y \in X$ and suppose $\|x - y\| < \varepsilon$. We observe that

$$|\|x\| - \|y\|| \leq \|x - y\| < \varepsilon$$

hence $\|\cdot\|$ is continuous at x, hence $\|\cdot\|$ is continuous. □

Definition. Let (x_n) be a sequence in $(X, \|\cdot\|)$. We call such a sequence **Cauchy** if

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N}^* : \forall n, m \geq N : \|x_n - x_m\| < \varepsilon$$

Theorem. *Let* $X = \mathbb{R}^p$*,* $p \in \mathbb{N}^*$*. Then every Cauchy sequence converges.*

Definition. We say that a normed space $(X, \|\cdot\|)$ is **complete** or **Banach** if for all sequences (x_n) in X we have

$$(x_n) \text{ Cauchy} \iff (x_n) \text{ converges}$$

Theorem. *Let I be a compact set in $\mathbb{R}$ (with respect to the standard topology $\mathcal{O}_{st}$). Then $C(I, \mathbb{R})$ is Banach.*

Definition.

$$l_2 := \left\{ (x_n) \text{ sequence in } \mathbb{R} \,\middle|\, \sum_{k=1}^{\infty} x_k \text{ converges} \right\}$$

$$\mathcal{L}_2(I, \mathbb{R}) := \left\{ f : I \to \mathbb{R} \,\middle|\, \int_I (f(t))^2 \, dt \text{ converges} \right\}$$

Claim. *Both of these spaces are Banach (at least when defined properly using Lesbegue integrals).*

Definition. Let X be a real or complex vector space. A map $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ is called an **inner product** over X if:

- $\forall x, y \in X : \langle x, y \rangle = \overline{\langle y, x \rangle}$
- $\forall \alpha \in F : \forall x, y, z \in X : \langle \alpha x + z, y \rangle = \alpha \langle x, y \rangle + \langle z, y \rangle$
- $\forall x \in X \setminus \{0_X\} : \langle x, x \rangle > 0$

We furthermore induce the map $\|\cdot\| : X \to \mathbb{R}$ such that

$$x \mapsto \sqrt{\langle x, x \rangle}$$

Lemma. *For all $x, y \in X$ and $x \perp y$ we have that*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2$$

Lemma.

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

Lemma. *The inner product is continuous in the first argument.*

Proof. Let $z, y \in X$, let $z_n \to z$ be a sequence in X. Observe that

$$|\langle z_n, y \rangle - \langle z, y \rangle| = |\langle z_n - z, y \rangle| \leq \|z_n - z\| \|y\| \to 0$$

Hence

$$\langle z_n, y \rangle \to \langle z, y \rangle$$

□

Claim. *For all $x, y \in X$ there exists uniquely $\alpha y, v \in X$ for some $\alpha \in F$ such that $v \perp y$ and $x = ay + v$.*

Proof.
- Case 1: $y = 0$, this is trivial
- Case 2: $y \neq 0$, observe that

$$\begin{aligned} \langle x - \alpha y, y \rangle = 0 &\iff \langle x, y \rangle - \alpha \langle y, y \rangle = 0 \\ &\iff \alpha = \frac{\langle x, y \rangle}{\|y\|^2} \end{aligned}$$

□

Theorem. *Let X be an inner product space. Let $x, y \in X$. Then*

- *If $y \neq 0_X$ then*
$$\min_{\alpha \in F} \|x - \alpha y\|^2 = \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2}$$
- *The Cauchy-Schwarz inequality*
- *The triangle inequality.*

Proof. Let $\tilde{\alpha} := \frac{\langle x, y \rangle}{\|y\|^2}$. Observe that

$$\begin{aligned} \|x - \alpha y\|^2 &= \|x - \tilde{\alpha} y - (\alpha - \tilde{\alpha}) y\|^2 \\ &= \|x - \tilde{\alpha} y\|^2 + \|(\alpha - \tilde{\alpha}) y\|^2 \\ &\geq \|x - \tilde{\alpha} y\|^2 \end{aligned}$$

Hence by definition we have the desired result. □

Definition. Let X be a normed space. Let $x \in X$. Let $V \leqslant X$ be a subspace. We say that some $\tilde{v} \in V$ is the **best approximation** of x if for all $v \in V$ we have that

$$\|x - v\| \geq \|x - \tilde{v}\|$$

Theorem. *Let X be an inner product space. Let $V \leqslant X$ be a subspace. Let $x \in X$. Then*

- *$\tilde{v} \in V$ is the unique best approximation of v if and only if $x - v \in V^{\perp}$*
- *If $\tilde{v}$ is a best approximation then $\|x - \tilde{v}\|^2 = \|x\|^2 - \|\tilde{v}\|^2$. In particular, $\|\tilde{v}\|^2 \leq \|x\|$.*

Proof. The second property is an easy corollary to the Pythagorean theorem, so we only prove the first statement. Let us first prove the $\Longleftarrow$ direction. Suppose indeed $x - \tilde{v} \in V^{\perp}$. Observe that for all $v \in V$

$$\begin{aligned} \|x - v\|^2 &= \|x - \tilde{v} - v + \tilde{v}\|^2 \\ &= \|x - \tilde{v}\|^2 + \|v - \tilde{v}\|^2 \\ &\geq \|x - \tilde{v}\|^2 \end{aligned}$$

Hence indeed $\tilde{v}$ is a best approximation. Consider that if $v \neq \tilde{v}$ then $\|v - \tilde{v}\|^2 > 0$ so we even have a strict inequality, which proves the uniqueness.
Now we prove $\Longrightarrow$ by contrapositive: suppose $x - \tilde{v} \notin V^{\perp}$, hence there exists $v \in V$ such that $\langle x - \tilde{v}, v\rangle = 0$. Observe that

$$\begin{aligned} \min_{\alpha \in F} \|x - \tilde{v} - \alpha v\| &= \|x - \tilde{v}\|^2 - \frac{|\langle x - \tilde{v}, v\rangle|^2}{\|v\|^2} \\ &< \|x - \tilde{v}\|^2 \end{aligned}$$

Hence in such case $\tilde{v}$ is not a best approximation. □

Definition. Let $(X, \langle\cdot\rangle)$ be an inner product space. X is called a **Hilbert space** if the induced norm (that is, $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$) on the space constitutes a Banach space, that is, the tuple $(X, \|\cdot\|)$ is Banach.

Theorem. *Let X be a Hilbert space. Let $e_1, \ldots$ be a (potentially infinite) sequence in X of orthonormal vectors. Let $x \in X$. Then the following statements are equivalent*

- $\forall k \in \mathbb{N}^* : \langle x, e_k\rangle = 0 \implies x = 0_X$
- $x = \sum_{k=1}^{\infty} \langle x, e_k\rangle e_k$
- $\|x\|^2 = \sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2$

Proof. Assume the first statement is true, that is, for all $k \in \mathbb{N}^*$ we have that $\langle x, e_k\rangle = 0$ implies that $x = 0_X$. Define the sequences

$$x_n := \sum_{k=1}^{n} \langle x, e_k\rangle e_k$$

$$\alpha_n := \sum_{k=1}^{n} |\langle x, e_k\rangle|^2$$

Observe that α_n is increasing and bounded above by $\|x\|^2$, hence α_n converges to some $\alpha \in \mathbb{R}$. Now let

$n, m \in \mathbb{N}^*$ and suppose $n > m$. Observe that

$$\begin{aligned}
\|x_n - x_m\|^2 &= \left\| \sum_{k=m+1}^{n} \langle x, e_k \rangle e_k \right\|^2 \\
&\leq \sum_{k=m+1}^{n} \|\langle x, e_k \rangle e_k\|^2 \\
&= \sum_{k=m+1}^{n} |\langle x, e_k \rangle|^2 \\
&= \alpha_n - \alpha_m \to 0 && \text{(because } \alpha_n \text{ is Cauchy)}
\end{aligned}$$

Hence x_n is Cauchy, and because of the assumption of a Hilbert space, x_n converges to some $\tilde{x} \in X$. Let $k \in \mathbb{N}^*$. Consider

$$\begin{aligned}
\langle x - \tilde{x}, e_k \rangle &= \langle x, e_k \rangle - \langle \tilde{x}, e_k \rangle \\
&= \langle x, e_k \rangle - \lim_{n \to \infty} \sum_{j=1}^{n} \langle x, e_j \rangle \langle e_j, e_k \rangle \\
&= \langle x, e_k \rangle - \lim_{n \to \infty} \langle x, e_k \rangle \\
&= \langle x, e_k \rangle - \langle x, e_k \rangle = 0
\end{aligned}$$

Hence by assumption $x - \tilde{x} = 0_X$, hence $x = \tilde{x}$.
The second property trivially implies the third, and the third property implies that if the inner products are 0 then $\|x\|^2 = 0 \implies \|x\| = 0 \implies x = 0_X$. □

2 Complex numbers and signals

Recall the complete factorization theorems:

Theorem. *Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree at most $n \in \mathbb{N}$. Suppose that f is not constant. Then there exists a $z_0 \in \mathbb{C}$ such that $f(z_0) = 0$.*

Theorem. *Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree at most $n \in \mathbb{N}$. Then there exist $c, z_1, \ldots z_n \in \mathbb{R}$ such that for all $z \in C$ we have*

$$f(z) = c \prod_{k=1}^{n} (z - z_k)$$

Proof. By induction. The base case is trivial. Let f now be a polynomial of degree at most n and suppose we know that for any other polynomial g of degree $n-1$ we can find a factorization as above. Observe that if the polynomial is constant the proof is also trivial, so we exclude this case. Then there exists a $z_n \in \mathbb{C}$ that is a root of f. We define a new polynomial, $j(h) := f(z_n + h)$. Observe that $j(0) = f(z_n) = 0$. But j is also a polynomial (there is a hidden induction here), hence we know that there exist $a_1, \ldots, a_n \in \mathbb{C}$ such that for all $h \in \mathbb{C}$ we have

$$j(h) = a_1 h + \ldots a_n h^n = h \underbrace{(a_1 + \ldots + a_n h^{n-1})}_{=: g(h)}$$

Then g is a polynomial of degree at most $n-1$, hence we find $c, z_1, \ldots, z_{n-1} \in \mathbb{C}$ that factorize g. Furthermore observe that

$$\begin{aligned} f(z) &= f(z_n + (z - z_n)) \\ &= j(z - z_n) \\ &= (z - z_n) \cdot g(z - z_n) \\ &= (z - z_n) c \prod_{k=1}^{n-1} (z - z_k) \\ &= c \prod_{k=1}^{n} (z - z_k) \end{aligned}$$

Which completes the proof. □

Definition. Alongside the definition of differentiability over general normed spaces or limits within metric spaces, we take the convention in this course to define the following.
Let $f : D \subseteq \mathbb{R} \to \mathbb{C}$ be a function, D open / of more than one point. Let $x \in D$. If the following limit exists, we call it the derivative of f at x at well and we say that f is then there also differentiable

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

Remark. One can evaluate and make the observation that

$$f'(t) = (\operatorname{Re} f)'(t) + i \cdot (\operatorname{Im} f)'(t)$$

Definition. Let $f : [a, b] \to \mathbb{C}$ be a function and suppose that $\operatorname{Re} f$ and $\operatorname{Im} f$ are integrable. Then we also say that f is integrable and define

$$\int_a^b f(t)\, dt := \int_a^b (\operatorname{Re} f)(t)\, dt + i \int_a^b (\operatorname{Im} f)(t)\, dt$$

Definition. Let $f : \mathbb{R} \to \mathbb{C}$ be a function. f is called **periodic** if there exists $C > 0$ such that for all $x \in \mathbb{R}$ we have $f(x + C) = f(x)$.

Definition. Let $A \geq 0$, $\omega \in \mathbb{R}$, $\phi \in \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(t) := A\cos(\omega t + \phi)$$

is called a **real harmonic signal**. We say that A is the **amplitude**, Ω is the (angular) **frequency**, and ϕ is the **initial phase**.

Definition. Let $C \in \mathbb{C}$, $\omega \in \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{C}$ given by

$$f(t) := C\exp(i\omega t)$$

is called a **complex harmonic signal**. We say that $|C|$ is the **amplitude**, $\arg(C)$ is the **initial phase**, and ω is the (angular) **frequency**.

Remark. Observe that harmonic signals are periodic. We can see this in the complex case and then prove a similar statement in the real case. Observe that

$$\exp(i\omega t) = \cos(\omega t) + i\sin(\omega t)$$

But then also

$$\exp(i(\omega t + 2\pi)) = \cos(\omega t + 2\pi) + i\sin(\omega t + 2\pi)$$

Hence if we have a complex harmonic f we have that $f(t) = f(t + \frac{2\pi}{\omega})$.
A similar observation in this regard is that the sum of harmonics with the same frequency is a harmonic with that frequency.

Definition. We define the following standard 'signals' as maps $\mathbb{R} \to \mathbb{C}$

- For some $a > 0$, the **rectangular pulse**

$$\mathrm{Rect}_a(t) := \begin{cases} 1 & \text{if } t \in [-\frac{a}{2}, \frac{a}{2}] \\ 0 & \text{otherwise} \end{cases}$$

- For some $a > 0$, the **triangular pulse**

$$\mathrm{Triang}_a(t) := \begin{cases} 1 - \frac{|t|}{a} & \text{if } t \in [-a, a] \\ 0 & \text{otherwise} \end{cases}$$

- The **step function**

$$\mathbb{1}(t) := \begin{cases} 1 & \text{if } t > 0 \\ \frac{1}{2} & \text{if } t = 0 \\ 0 & \text{if } t < 0 \end{cases}$$

- The **sinc function**

$$\mathrm{sinc}(t) := \begin{cases} \frac{\sin(t)}{t} & \text{if } t \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

  (the continuous extension of $t \mapsto \frac{\sin(t)}{t}$)

3 Delta function and convolutions

We have seen rectangular pulses. One way to define them is as such: for $n \in \mathbb{N}^*$, define a map $R_n : \mathbb{R} \to \mathbb{R}$ given by

$$t \mapsto \begin{cases} n & \text{if } -\frac{1}{2n} \leq t \leq \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases}$$

Clearly, we are dealing with a rectangle of width $\frac{1}{n}$ and height n, so

$$\int_{\mathbb{R}} R_n(t)\, dt = 1$$

for all $n \in \mathbb{N}^*$. Clearly, for all $t \neq 0$, $R_n(t)$ converges to 0 as $n \to \infty$, however, at zero, $R_n(t)$ diverges to ∞ as $n \to \infty$. Despite this, we make the very dangerous assertion that such a limit does exist, heuristically making a new function $\delta : \mathbb{R} \to \mathbb{R}$ 'defined by'

$$\delta(t) := \lim_{n\to\infty} R_n(t)$$

Now, even though this function does not converge uniformly at all to any type of regular function, we again make the assumption that it is okay to interchange limits, as such:

$$\int_{\mathbb{R}} \delta(t)\, dt = \int_{\mathbb{R}} \lim_{n\to\infty} R_n(t)\, dt = \lim_{n\to\infty} \int_{\mathbb{R}} R_n(t)\, dt = 1$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Observe that for all $t \neq 0$ we have $\delta(t) = 0$, hence for all $b, t \in \mathbb{R}$ we have that

$$f(t)\delta(t-b) = f(b)\delta(t-b)$$

This gives us the interesting result that

$$\int_{\mathbb{R}} f(t)\delta(t-b)\, dt = f(b) \int_{\mathbb{R}} \delta(t-b)\, dt = f(b)$$

In particular,

$$\int_{\mathbb{R}} f(t)\delta(t)\, dt = f(0)$$

Now, we would like to observe what happens if we let $a \in \mathbb{R} \setminus \{0\}$ and consider for some $t \in \mathbb{R}$, $\delta(at)$. Well, going back to the limit definition,

$$\lim_{n\to\infty} R_n(at) = \lim_{n\to\infty} \frac{1}{|a|} R_{\lceil |a|n \rceil}(t) = \frac{1}{|a|}\delta(t)$$

Furthermore observe that for some $t \in \mathbb{R}$

$$\int_{-\infty}^{t} f(\xi)\, d\xi = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0 \end{cases} = \mathbb{1}(t)$$

So in a way (cry) $\mathbb{1}' = \delta$.

Example. Consider $|\cdot| : \mathbb{R} \to \mathbb{R}$. We can write for all $t \in \mathbb{R}$

$$|t| = t(\mathbb{1}(t) - \mathbb{1}(-t))$$

Then

$$\begin{aligned} |\cdot|'(t) &= \mathbb{1}(t) - \mathbb{1}(-t) + t(\delta(t) + \delta(-t)) \\ &= \mathbb{1}(t) - \mathbb{1}(-t) + t(\delta(t) - \delta(t)) \\ &= \mathbb{1}(t) - \mathbb{1}(-t) = \operatorname{sgn}(t) \end{aligned}$$

This was already known for $t \neq 0$.

Definition. Let $f, g : \mathbb{R} \to \mathbb{R}$ be improperly integrable functions. If it exists, we define a new function $f * g : \mathbb{R} \to \mathbb{R}$ given by

$$(f * g)(t) := \int_{\mathbb{R}} f(\xi)g(t-\xi)\, d\xi$$

Claim. *For all convolvable functions f, g, $f * g = g * f$.*

Proof. Let $t \in \mathbb{R}$. Observe that

$$\begin{aligned}(g * f)(t) &= \int_{-\infty}^{\infty} g(\xi)f(t-\xi)\, d\xi \\ &= \int_{\infty}^{-\infty} -g(t-\xi)f(\xi)\, d\xi \\ &= \int_{-\infty}^{\infty} f(\xi)g(t-\xi)\, d\xi \\ &= (f * g)(t)\end{aligned}$$

□

4 Fourier theory

Remark. The upshot of this section is that, given a signal $f : \mathbb{R} \to \mathbb{C}$ such that f is $T \in \mathbb{R}$ periodic, then there exists a sequence $(f_k)_{k\in\mathbb{Z}}$ in $\mathbb{C}$ such that for (almost) every $t \in \mathbb{R}$, we have that

$$f(t) = \sum_{k=-\infty}^{\infty} f_k \exp\left(ik\frac{2\pi}{T}t\right)$$

Definition. Let $f : \mathbb{R} \to \mathbb{C}$ be a T-periodic signal, $T \in \mathbb{R}$. We define a quantity called the **fundamental frequency** of f, ω_0, by

$$\omega_0 := \frac{2\pi}{T}$$

Theorem. *Let $T > 0$. Let $(e_k)_{k\in\mathbb{Z}}$ be a sequence in $\mathcal{L}^2([0,T];\mathbb{C})$ defined by*

$$e_k : \mathbb{R} \to \mathbb{C}$$

$$e_k(t) := \exp(i\omega_0 kt)$$

Then (e_k) is complete and orthonormal (w.r.t. the standard inner product on $\mathcal{L}^2$ spaces).

Proof.
- Recall the standard inner product: for two $f, g \in \mathcal{L}^2([0,T];\mathbb{C})$ we have that

$$\langle f, g\rangle := \frac{1}{T}\int_0^T f(t)\overline{g(t)}\,dt$$

Observe that for some $k \in \mathbb{Z}$ we have

$$\|e_k\|^2 = \langle e_k, e_k\rangle = \frac{1}{T}\int_0^T \exp(i\omega_0 kt)\overline{\exp(i\omega_0 kt)}\,dt = \frac{T}{T} = 1$$

Now let $j \in \mathbb{Z}$ such that $j \neq k$. Observe that

$$\begin{aligned}\langle e_k, e_j\rangle &= \frac{1}{T}\int_0^T \exp(i\omega_0 kt)\overline{\exp(i\omega_0 jt)}\,dt \\ &= \frac{1}{T}\int_0^T \exp(i\omega_0 kt)\exp(-i\omega_0 jt)\,dt \\ &= \frac{1}{T}\int_0^T \exp(i\omega_0(k-j)t)\,dt \\ &= \frac{1}{T}\frac{\exp(i\omega_0(k-j)t)}{i\omega_0(k-j)}\Bigg|_0^T \\ &= \frac{1}{T}\frac{\exp(i2\pi(k-j))-1}{i\omega_0(k-j)} = 0\end{aligned}$$

- Completeness will be omitted because of the lack of a rigorous proof in the lectures.

□

Corollary. *Every T-periodic signal equals its Fourier series (strictly speaking, the corresponding equivalence classes in $\mathcal{L}^2$ are equal, but the functions are not equal at finitely many points).*

Definition. (depending on where you come from, this is a recall). Let $(a_k)_{k\in\mathbb{Z}}$ be a sequence in $\mathbb{C}$. We say that the series

$$\sum_{k=-\infty}^{\infty} a_k$$

converges if there exists an $L \in \mathbb{C}$ such that

$$L = \lim_{n\to\infty}\sum_{k=-N}^{N} a_k$$

Theorem. *Let $f : \mathbb{R} \to \mathbb{C}$ be a T-periodic signal. Let $(f_k)_{k\in\mathbb{Z}}$ be its Fourier coefficients, that is,*

$$f_k := \langle f, e_k \rangle$$

Suppose that the following series converges (in $\mathbb{R}$)

$$\sum_{k=-\infty}^{\infty} |f_k|$$

Then the Fourier series of f converges at all points, and the Fourier series is continuous.

Proof. Let $t \in \mathbb{R}$. Clearly,

$$\sum_{k=-\infty}^{\infty} \left| f_k \exp\left(ik\frac{2\pi}{T}t \right) \right| = \sum_{k=-\infty}^{\infty} |f_k|$$

which converges, hence the Fourier series absolutely converges. But if a series of functions converges absolutely at every point, the series itself converges uniformly to some limit function. Now, the partial functions are all continuous, hence the limit function is continuous. □

Definition. Let $f : \mathbb{R} \to \mathbb{C}$ be a regulated function, that is, the right and left limits exist at every point. We define for such function and for every $t \in \mathbb{R}$ the following numbers

$$f(t^+) := \lim_{x \to t^+} f(x) \quad , \quad f(t^-) := \lim_{x \to t^-} f(x)$$

Theorem. *Let $f : \mathbb{R} \to \mathbb{C}$ be T-periodic and suppose f is* ***piecewise smooth****, that is, there exists a partition of $\mathbb{R}$ such that the function, when restricted to any open interval of the partition, is smooth. Let $\tilde{f}$ be the Fourier series of f. Let $t \in \mathbb{R}$. Then*

$$\frac{f(t^+) + f(t^-)}{2} = \tilde{f}(t)$$

If additionally f is continuous, then $f = \tilde{f}$ (now finally in the classical sense instead of some arcane make-believe wishful thinking $\mathcal{L}^2$ sense).

Lemma. *Let $f : \mathbb{R} \to \mathbb{C}$ be piecewise smooth on $[a, b]$ for some $a < b$ real numbers. Then*

$$\lim_{\omega \to \infty} \int_a^b f(t) \exp(i\omega t)\, dt = 0$$

Proof. Suppose that f is continuously differentiable. Then by parts we find for some $\omega > 0$ that

$$\int_a^b f(t) \exp(i\omega t)\, dt = \frac{1}{i\omega} \left(f(t) \exp(i\omega t) - \int_a^b f'(t) \exp(i\omega t)\, dt \right)$$

By the triangle inequality for integrals,

$$\left| \int_a^b f(t) \exp(i\omega t)\, dt \right| \le \frac{1}{\omega} \left(|f(t)| - \int_a^b |f'(t)|\, dt \right) \to 0 \qquad (\omega \to \infty)$$

If f is not continuously differentiable, we find a partition of $[a, b]$ such that f restricted to open intervals in the partition is smooth, e.g. also continuously differentiable, and integrable on closed intervals. By linearity of the integral we have the desired result. □

Remark. Observe that if we have a T-periodic signal $f : \mathbb{R} \to \mathbb{R}$ (real-valued), then the Fourier coefficients $(f_k)_{k\in\mathbb{Z}}$ are symmetric in a way, namely that for all $k \in \mathbb{Z}$ we have that

$$f_{-k} = \overline{f_k}$$

We can see this easily:

$$\begin{aligned}
f_{-k} &= \langle f, e_{-k} \rangle \\
&= \frac{1}{T} \int_0^T f(t) \overline{\exp(-ik\omega_0 t)}\, dt \\
&= \frac{1}{T} \int_0^T \overline{f(t)} \exp(ik\omega_0 t)\, dt \\
&= \overline{\frac{1}{T} \int_0^T f(t) \exp(-ik\omega_0 t)\, dt} \\
&= \overline{f_k}
\end{aligned}$$

Observe that if for some $t \in \mathbb{R}$ the Fourier series in f converges, then we have that

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} f_k \exp(ik\omega_0 t) &= \sum_{k=-\infty}^{0} f_k \exp(ik\omega_0 t) + \sum_{k=1}^{\infty} f_k \exp(ik\omega_0 t) \\
&= f_0 + \sum_{k=-\infty}^{-1} f_k \exp(ik\omega_0 t) + \sum_{k=1}^{\infty} f_k \exp(ik\omega_0 t) \\
&= f_0 + \sum_{k=1}^{\infty} f_{-k} \exp(-ik\omega_0 t) + \sum_{k=1}^{\infty} f_k \exp(ik\omega_0 t) \\
&= f_0 + \sum_{k=1}^{\infty} \overline{f_k \exp(ik\omega_0 t)} + \sum_{k=1}^{\infty} f_k \exp(ik\omega_0 t) \\
&= f_0 + \sum_{k=1}^{\infty} \overline{f_k \exp(ik\omega_0 t)} + f_k \exp(ik\omega_0 t) \\
&= f_0 + \sum_{k=1}^{\infty} \operatorname{Re} 2 f_k \exp(ik\omega_0 t) \\
&= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)
\end{aligned}$$

where

$$a_k := 2 \operatorname{Re} f_k \quad , \quad b_k := -2 \operatorname{Im} f_k$$

Definition. Let $f : \mathbb{R} \to \mathbb{C}$ be a signal. We say that f is an **odd signal** if f is an odd function, that is, for all $t \in \mathbb{R}$ we have

$$f(-t) = -f(t)$$

We say that f is an **even signal** if for all $t \in \mathbb{R}$ we have

$$f(-t) = f(t)$$

Remark. If f is both even and odd, we have that for all $t \in \mathbb{R}$, $-f(t) = f(t)$, hence f must be constant zero.

5 Fourier transform

Earlier, we saw how to break up periodic signals into a series of complex harmonics with integer frequencies. We can now extend this idea to all real frequencies, so the analogous idea is to *integrate* over $\mathbb{R}$, adding all complex signals $t \mapsto \exp(i\omega t)$ for $\omega \in \mathbb{R}$. The coefficient for a certain harmonic should now be assigned for all $\omega \in \mathbb{R}$, so we have a function $\hat{f} : \mathbb{R} \to \mathbb{C}$.
If this exists, $\hat{f}$ is called the **Fourier transform** of $f : \mathbb{R} \to \mathbb{C}$, such that for all $t \in \mathbb{R}$ we have

$$f(t) = \int_{\mathbb{R}} \hat{f}(\omega) \exp(i\omega t)\, dt$$

For convenience throughout the next few lectures / chapters, we only look at piecewise smooth signals, and make the assumption that points of discontinuity are redefined to be the arithmetic mean of the left and right hand limits, e.g. for all $t \in \mathbb{R}$ we have

$$f(t) = \frac{f(t^+) + f(t^-)}{2}$$

5.1 Fourier integral theorem

We first collect some lemma's to work towards a theorem that states when we can write f as above.

Lemma (Riemann-Lebesgue)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be an absolutely integrable and piecewise smooth signal. Then*

$$\lim_{\omega \to \infty} \left| \int_{\mathbb{R}} f(t) \exp(-i\omega t)\, dt \right| = 0$$

Proof. We present a proof that differs from the lecture notes. Consider that if f is absolutely integrable and piecewise smooth, it is continuous at most points. Hence it is a regulated function, which means it is the uniform limit of step functions.
We now show the result for a sequence of step functions, $(f_n)_{n\in\mathbb{N}^*}$ that uniformly converges to f, $f_n : \mathbb{R} \to \mathbb{C}$. Consider that any f_n can be written as a series of indicator functions on intervals, and because of uniform convergence and linearity of the integral it suffices to show the result for only such an indicator.
Well, consider thus when $f = z1_{[a,b]}$, $a < b$ real numbers, $z \in \mathbb{C}$ arbitrary. Observe that

$$\begin{aligned} \lim_{\omega \to \infty} \left| \int_{\mathbb{R}} f(t) \exp(-i\omega t)\, dt \right| &= \lim_{\omega \to \infty} \left| z \int_a^b \exp(-i\omega t)\, dt \right| \\ &= \lim_{\omega \to \infty} \left| z \frac{i}{\omega} \exp(-i\omega t) \Big|_a^b \right| = 0 \end{aligned}$$

Now to complete the proof, recall that there exists for every $n \in \mathbb{N}^*$ a sequence $(z_k^n)_{k\in\mathbb{N}^*}$, $z_k \in \mathbb{C}$ and a sequence of intervals $(I_k^n)_{k\in\mathbb{N}^*}$, such that

$$f_n = \sum_{k=1}^{\infty} z_k^n 1_{I_k^n}$$

Then

$$\begin{aligned} \lim_{\omega \to \infty} \left| \int_{\mathbb{R}} f(t) \exp(-i\omega t)\, dt \right| &= \lim_{n \to \infty} \lim_{\omega \to \infty} \left| \sum_{k=1}^{\infty} z_k^n \int_{I_k^n} \exp(-i\omega t)\, dt \right| \\ &= \lim_{n \to \infty} 0 = 0 \end{aligned}$$

□

Lemma. *Let $f : \mathbb{R} \to \mathbb{C}$ be an absolutely integrable and piecewise smooth signal. Then*

$$\lim_{a \to \infty} \int_{\mathbb{R}} f(t) a \operatorname{sinc}(at)\, dt = \pi f(0) = \pi \frac{f(0^+) + f(0^-)}{2}$$

Theorem. *Let $f : \mathbb{R} \to \mathbb{C}$ be an absolutely integrable and piecewise smooth signal. Then for all $t \in \mathbb{R}$*

$$f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\omega) \exp(i\omega t)\, d\omega$$

where for all $\omega \in \mathbb{R}$ we have

$$\hat{f}(\omega) := \int_{\mathbb{R}} f(t) \exp(-i\omega t)\, dt$$

Proof. Let $t \in \mathbb{R}$.

$$\begin{aligned}
\frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\omega) \exp(i\omega t)\, d\omega &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(s) \exp(-i\omega s) \exp(i\omega t)\, ds\, d\omega \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(s) \exp(i\omega(t-s))\, ds\, d\omega \\
&= \lim_{x\to\infty} \frac{1}{2\pi} \int_{\mathbb{R}} \int_{-x}^{x} f(s) \exp(i\omega(t-s))\, d\omega\, ds \\
&= \lim_{x\to\infty} \frac{1}{2\pi} \int_{\mathbb{R}} f(s) \frac{-i}{t-s} \exp(i\omega(t-s)) \Big|_{-x}^{x} ds \\
&= \lim_{x\to\infty} \frac{1}{2\pi} \int_{\mathbb{R}} f(s) \frac{-i}{t-s} \exp(i\omega(t-s)) \Big|_{-x}^{x} ds
\end{aligned}$$

□

5.2 Fourier transform

The functions f and $\hat{f}$ are related, one can recover the other. Furthermore, a scaled version of the Fourier transform of $\hat{f}$ is f. We can then define the **Fourier transform** itself instead as a map that maps piecewise smooth functions to piecewise smooth functions. We can define a set $C^{1/2}$, for example, as the set of piecewise smooth functions with redefined points of discontinuity as seen before. Then

$$\begin{gathered}
\mathfrak{F} : C^{1/2} \to C^{1/2} \\
f \mapsto \left(\omega \mapsto \int_{\mathbb{R}} f(t) \exp(-i\omega t)\, dt \right)
\end{gathered}$$

is called the **Fourier transform**. As we have seen, this map is bijective, where

$$\begin{gathered}
\mathfrak{F}^{-1} : C^{1/2} \to C^{1/2} \\
\hat{f} \mapsto \left(t \mapsto \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\omega) \exp(i\omega t)\, d\omega \right)
\end{gathered}$$

Remark. There is some silly notation out there: instead of writing $\mathfrak{F}(f)$ we write $\mathfrak{F}\{f\}$.

Remark. Suppose $f : \mathbb{R} \to \mathbb{C}$ is differentiable and $f, f' \in \mathcal{L}^1$, that is, they are absolutely integrable on $\mathbb{R}$. Then clearly because f' is absolutely integrable, f' goes to zero at $\pm\infty$. Then for $t \in \mathbb{R}$

$$\begin{aligned}
\mathfrak{F}\{f'\}(\omega) &= \int_{\mathbb{R}} f'(t) \exp(-i\omega t)\, dt \\
&= f(t) \exp(-i\omega t) \Big|_{-\infty}^{\infty} + i\omega \int_{\mathbb{R}} f(t) \exp(-i\omega t) \\
&= i\omega \mathfrak{F}\{f\}(\omega)
\end{aligned}$$

Remark. Suppose $f : \mathbb{R} \to \mathbb{C}$ is integrable and continuous, let $F : \mathbb{R} \to \mathbb{C}$ be an antiderivative. Suppose that $\lim_{t\to\pm\infty} f(t) = 0$. Observe that

$$\begin{aligned}
\mathfrak{F}\{F\}(\omega) &= \int_{\mathbb{R}} F(t) \exp(-i\omega t)\, dt \\
&= -f(t) \frac{1}{i\omega} \exp(-i\omega t) \Big|_{-\infty}^{\infty} + \frac{1}{i\omega} \int_{\mathbb{R}} f(t) \exp(-i\omega t)\, dt \\
&= \frac{1}{i\omega} \mathfrak{F}\{f\}(\omega) && \text{(if } \omega \neq 0\text{)}
\end{aligned}$$

Remark. Recall that for all $t \in \mathbb{R}$

$$\cos(t) := \frac{\exp(it) + \exp(-it)}{2}$$

Observe that

$$\begin{aligned}
\mathfrak{F}\{f\cos(\omega_0\cdot)\}(\omega) &= \int_{\mathbb{R}} f(t)\cos(\omega_0 t)\exp(-i\omega t)\,dt \\
&= \int_{\mathbb{R}} f(t)\frac{\exp(i\omega_0 t) + \exp(-i\omega_0 t)}{2}\exp(-i\omega t)\,dt \\
&= \int_{\mathbb{R}} f(t)\frac{\exp(i\omega_0 t)}{2}\exp(-i\omega t)\,dt + \int_{\mathbb{R}} f(t)\frac{\exp(-i\omega_0 t)}{2}\exp(-i\omega t)\,dt \\
&= \frac{1}{2}\int_{\mathbb{R}} f(t)\exp(-i(\omega-\omega_0)t)\,dt + \frac{1}{2}\int_{\mathbb{R}} f(t)\exp(-i(\omega+\omega_0)t)\,dt \\
&= \frac{1}{2}(\mathfrak{F}\{f\}(\omega-\omega_0) + \mathfrak{F}\{f\}(\omega+\omega_0))
\end{aligned}$$

Several properties of $\mathfrak{F}$ are:

- Let $f, g \in C^{1/2}$, $a, b \in \mathbb{C}$. Then $\mathfrak{F}\{af + bg\} = a\mathfrak{F}\{f\} + b\mathfrak{F}\{g\}$.
- $\mathfrak{F}\{\mathfrak{F}\{f\}\} = 2\pi f(-\bullet)$
- $\mathfrak{F}(f^*) = \mathfrak{F}\{f\}^*(-\bullet)$, where f^* denotes the complex conjugate function, given by

$$f^*(t) := \operatorname{Re} f(t) - i\operatorname{Im} f(t)$$

- $\mathfrak{F}\{t \mapsto f(at)\} = \omega \mapsto \frac{1}{|a|}\mathfrak{F}\{f\}\left(\frac{\omega}{a}\right)$
- For all $a \in \mathbb{R}$, $\mathfrak{F}\{t \mapsto f(t-a)\} = \omega \mapsto \mathfrak{F}\{f\}(\omega)\exp(-ia\omega)$
- $\mathfrak{F}\{f\exp(i\omega_0\cdot)\} = \omega \mapsto \mathfrak{F}\{f\}(\omega-\omega_0)$
- $\mathfrak{F}\{f\cos(\omega_0\cdot)\}(\omega) = \frac{1}{2}(\mathfrak{F}\{f\}(\omega-\omega_0) + \mathfrak{F}\{f\}(\omega+\omega_0))$
- $\mathfrak{F}\{f'\} = \omega \mapsto i\omega\mathfrak{F}\{f\}(\omega)$.
- $\mathfrak{F}\{F\} = \frac{1}{i\omega}\mathfrak{F}\{f\}$
- $\mathfrak{F}\{t \mapsto -itf(t)\} = \mathfrak{F}\{f\}'$

Example. We can use the (inverse) Fourier transform to compute the Gaussian integral, for example. Consider

$$\begin{aligned}
f : \mathbb{R} &\to \mathbb{R} \\
t &\mapsto \exp(-t^2)
\end{aligned}$$

Observe that its Fourier transform $\hat{f}$ is

$$\hat{f}(\omega) := \int_{\mathbb{R}} f(t)\exp(-i\omega t)\,dt = \int_{\mathbb{R}} f(t)\cos(\omega t)\,dt - i\int_{\mathbb{R}} f(t)\sin(\omega t)\,dt = \int_{\mathbb{R}} \exp(-t^2)\cos(\omega t)\,dt$$

Observe that

$$\begin{aligned}
\hat{f}'(\omega) &= -\int_{\mathbb{R}} t\exp(-t^2)\sin(\omega t)\,dt \\
&= \frac{1}{2}\exp(-t^2)\sin(\omega t)\Big|_{-\infty}^{\infty} - \frac{\omega}{2}\int_{\mathbb{R}} \exp(-t^2)\cos(\omega t)\,dt \\
&= -\frac{\omega}{2}\hat{f}(\omega)
\end{aligned}$$

Hence there exists $z \in \mathbb{C}$ such that

$$\hat{f}(\omega) = z \exp\left(-\frac{\omega^2}{4}\right)$$

If we let $\omega = 0$ then $z = \int_{\mathbb{R}} f(t)\,dt$. But then

$$f(0) = \frac{z}{2\pi} \int_{\mathbb{R}} \exp\left(-\frac{\omega^2}{4}\right) d\omega = \frac{z^2}{\pi} = 1$$

Hence

$$\int_{\mathbb{R}} \exp(-t^2)\,dt = \sqrt{\pi}$$

5.3 Transform of convolution

Theorem. *Let $f, g : \mathbb{R} \to \mathbb{C}$ be absolutely integrable, piecewise smooth, discontinuities replaced, and convolvable. Then*

$$\mathfrak{F}\{f * g\} = \mathfrak{F}\{f\}\mathfrak{F}\{g\}$$

Proof. Let $\omega \in \mathbb{R}$. Observe that

$$\begin{aligned}
\mathfrak{F}\{f * g\}(\omega) &= \int_{\mathbb{R}} (f * g)(t) \exp(-i\omega t)\,dt \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(\tau) g(t - \tau) \exp(-i\omega t)\,d\tau\,dt \\
&= \int_{\mathbb{R}} f(\tau) \mathfrak{F}\{t \mapsto g(t - \tau)\}(\omega)\,d\tau \\
&= \int_{\mathbb{R}} f(\tau) \mathfrak{F}\{g\}(\omega) \exp(-i\omega\tau)\,d\tau \\
&= \mathfrak{F}\{f\}(\omega)\mathfrak{F}\{g\}(\omega)
\end{aligned}$$

□

Corollary. *Let $f, g : \mathbb{R} \to \mathbb{C}$ be absolutely integrable, piecewise smooth, discontinuities replaced, and convolvable. Then*

$$fg = \frac{1}{2\pi}(\mathfrak{F}\{f\} * \mathfrak{F}\{g\})$$

Corollary. *Let $f : \mathbb{R} \to \mathbb{C}$ be absolutely integrable, piecewise smooth, discontinuities replaced. Recall the definition of its energy content:*

$$E_f := \int_{\mathbb{R}} |f(t)|^2\,dt$$

Suppose that E_f converges (in $\mathbb{R}$). Then

$$E_f = \frac{1}{2\pi} \int_{\mathbb{R}} |\mathfrak{F}\{f\}(\omega)|^2\,d\omega$$

6 Laplace transform

6.1 Definition

Definition. Let $f : \mathbb{R} \to \mathbb{C}$ be a signal. Let $I_s : [0, \infty) \to \mathbb{C}$ be a function for some $s \in \mathbb{C}$ such that $t \mapsto f(t)\exp(-st)$. Let

$$D := \{s \in \mathbb{C} \mid I_s \text{ is improperly integrable}\}$$

We then define the **Laplace transform** of f, $F : D \to \mathbb{C}$, by

$$F(s) := \int_0^\infty I_s(t)\, dt = \int_0^\infty f(t)\exp(-st)\, dt$$

We also write: $\mathcal{L}\{f\} := F$.

Remark. Under this definition, we can recall that if f is a **causal** signal, that is, it is zero for negative time ($\forall t < 0 : f(t) = 0$), then its Fourier transform can be related to the Laplace transform by

$$\mathfrak{F}\{f\}(\omega) = \mathcal{L}\{f\}(i\omega)$$

Theorem. *There exists an $\alpha \in \mathbb{R}$ such that the set D in the above definition satisfies*

$$\{s \in \mathbb{C} \mid \operatorname{Re} s > \alpha\} \subseteq D$$

or $D = \varnothing$.

Proof. By cases, if $D = \varnothing$ we are done. So now suppose $D \neq \varnothing$ and $z \in D$, so I_z is improperly integrable. Let $s \in \mathbb{C}$ and suppose $\operatorname{Re} s > \operatorname{Re} z$, e.g. it is in the above set. We shall show that I_s is also improperly integrable. Let F be an antiderivative of I_z. Its limit at infinity equals the improper integral of I_z. Hence, F is bounded on an interval $[c, \infty)$ for some $c \in \mathbb{R}$. Redefine / translate F so it instead is bounded on $[0, \infty)$. Observe that

$$\begin{aligned}
\int_0^\infty I_s(t)\, dt &= \int_0^\infty f(t)\exp(-st)\, dt \\
&= \int_0^\infty f(t)\exp(-(s-z)t)\exp(-zt)\, dt \\
&= \int_0^\infty \exp(-(s-z)t) I_z(t)\, dt \\
&= \exp(-(s-z)t)F(t)\Big|_0^\infty + (s-z)\int_0^\infty \exp(-(s-z)t)F(t)\, dt
\end{aligned}$$

Now the left part converges to some real number and the right side absolutely converges: Let $a := s - z$, observe that $\operatorname{Re} a = \operatorname{Re} s - \operatorname{Re} z > 0$.

$$\begin{aligned}
\left|\int_0^\infty \exp(-at)F(t)\right| dt &\leq \int_0^\infty |\exp(-at)F(t)|\, dt \\
&\leq \sup_{t\in[0,\infty)} |F(t)| \int_0^\infty |\exp(-\operatorname{Re} at)||\exp(-i \operatorname{Im} at)|\, dt \\
&= \sup_{t\in[0,\infty)} |F(t)| \int_0^\infty \exp(-\operatorname{Re} at)\, dt \\
&= -\sup_{t\in[0,\infty)} |F(t)| \frac{1}{\operatorname{Re} a} \exp(-\operatorname{Re} at)\Big|_0^\infty \\
&= \sup_{t\in[0,\infty)} |F(t)| \frac{1}{\operatorname{Re} a}
\end{aligned}$$

□

Definition. Let $f : \mathbb{R} \to \mathbb{C}$ be a signal. Let $D \subseteq \mathbb{C}$ be the region where the Laplace transform of f exists, like in the definitions above. Then we define the **abscissa of convergence** of the Laplace transform of f, $\alpha \in \mathbb{R} \cup \{-\infty, \infty\}$, by:

$$\alpha := \inf\{\alpha \in \mathbb{R} \mid \{s \in \mathbb{C} \mid \operatorname{Re} s > \alpha\} \subseteq D\}$$

where we use the convention that the infimum of an unbounded set is $-\infty$, and the infimum of an empty set is ∞. The existence of such α is thus guaranteed.

Example. Let $f = \mathbb{1}$. Consider its Laplace transform

$$\begin{aligned}\mathcal{L}\{\mathbb{1}\}(s) &= \int_0^\infty \exp(-st)\,dt \\ &= \frac{\exp(-st)}{s}\bigg|_0^\infty\end{aligned}$$

which converges only if $\operatorname{Re} s > 0$, hence $\alpha = 0$.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) := \exp(-x^2)$. Clearly, its Laplace transform exists at $s = 0$, where its value is a Gaussian. It also hence exists along the imaginary axis, just being half the Fourier transform. But its Laplace transform exists even for arbitrarily negative s. Hence, $\alpha = -\infty$.

Theorem. *Let $f, g : \mathbb{R} \to \mathbb{C}$ be causal signals that are piecewise smooth. For such choices, the Laplace transform is injective, that is*

$$\mathcal{L}\{f\} = \mathcal{L}\{g\} \implies f = g$$

at least, in the sense that their difference in norm is 0 (equal almost everywhere).

Corollary. *The Laplace transform, as a map from $\mathcal{L}^2((0,\infty);\mathbb{C})$ onto certain complex analytic functions, is bijective, that is, in most cases, the Laplace transform is invertible.*

6.2 The return of the delta functions

Consider that the delta function obviously in the standard notion does not have a Laplace transform, because it is not integrable, I suppose. But we can use our black magic voodoo:

$$\begin{aligned}\mathcal{L}\{\delta\}(s) &= \int_0^\infty \delta(t)\exp(-st)\,dt \\ &= \int_0^\infty \delta(t)\,dt = 1\end{aligned}$$

Or for a shifted delta, namely, $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ given by $f(t) := \delta(t - b)$ for $b \in \mathbb{R}$, then

$$\begin{aligned}\mathcal{L}\{f\}(s) &= \int_0^\infty \delta(t-b)\exp(-st)\,dt \\ &= \exp(-sb)\int_0^\infty \delta(t-b)\,dt \\ &= \begin{cases}\exp(-sb) & \text{if } b \geq 0 \\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

This is all under the assumption that the magical integral of the delta function is continuous, that is,

$$\int_0^\infty \delta(x)\,dx = \lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^\infty \delta(x)\,dx = 1$$

6.3 Properties

Theorem. *Let $f, g : \mathbb{R} \to \mathbb{C}$ be signals. Let $\alpha, \beta \in \mathbb{R}$ be the abscissae of convergence of their respective Laplace transforms. Then the following properties hold:*

- *For all $a, b \in \mathbb{C}$, $\mathcal{L}\{af + bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}$*
- *For all $a > 0$, $\mathcal{L}\{f(a\cdot)\} = \frac{1}{a}\mathcal{L}\{f\}\left(\frac{\cdot}{a}\right)$*
- *More...*

Theorem. *Let $f : \mathbb{R} \to \mathbb{C}$ be exponentially bounded and integrable, that is, there exists $a \in \mathbb{R}$, $C > 0$ such that for all $t \in \mathbb{R}$ we have $|f(t)| \leq C\exp(at)$. Let $F : \mathbb{C} \to \mathbb{C}$ be the Laplace transform of f. Then*

$$\lim_{|s| \to \infty} F(s) = 0$$

Proof. Let $s \in \mathbb{C}$ and suppose $\operatorname{Re} s > a$. Then

$$\begin{aligned}
|F(s)| &= \left| \int_0^\infty f(t)\exp(-st)\,dt \right| \\
&\leq \int_0^\infty |f(t)\exp(-st)|\,dt \\
&\leq C\int_0^\infty \exp((a - \operatorname{Re} s)t)\,dt \\
&= -\frac{C}{a - \operatorname{Re} s} \overset{|s|\to\infty}{\to} 0
\end{aligned}$$

□

Theorem. *Let $f : [0, \infty) \to \mathbb{C}$ be exponentially bounded and piecewise smooth. Let F be its Laplace transform Then*

$$\lim_{s\to\infty} sF(s) = \lim_{x \to 0^+} f(x)$$

Proof. Let $A \in [0, \infty)$. Let $g_n : [0, A] \to \mathbb{C}$ be a sequence of functions given by

$$g_n(t) := f\left(\frac{t}{n}\right)\exp(-t)$$

Furthermore, let $g : [0, A]$ be its pointwise limit, that is

$$g(t) := f(0^+)\exp(-t)$$

We would like to show that g_n converges to g uniformly. Because of uniform continuity on a compact set of exp, it suffices to show that $h_n(t) := f\left(\frac{t}{n}\right)$ converges uniformly to $f(0^+)$. But, being defined over a compact set, it attains a maximum and is hence bounded, which means it converges uniformly.

$$\begin{aligned}
\lim_{s\to\infty} \int_0^\infty f(t)\exp(-st)s\,dt &= \lim_{s\to\infty}\int_0^\infty f\left(\frac{t}{s}\right)\exp(-t)\,dt \\
&= \lim_{A\to\infty}\lim_{s\to\infty}\left(\int_0^A f\left(\frac{t}{s}\right)\exp(-t)\,dt + \int_A^\infty f\left(\frac{t}{s}\right)\exp(-t)\,dt\right) \\
&= \lim_{A\to\infty}\left(f(0^+) - f(0^+)\exp(-A) + C\exp(-A)\right) \\
&= f(0^+)
\end{aligned}$$

In the second to last step we assumed that f is bounded by some number, which allows us to conclude it integrates to $C\exp(-A)$ for some $C \in \mathbb{R}$. Now suppose f is merely exponentially bounded, then there exists $C > 0$ and $a \in \mathbb{R}$ such that for all $t \in [0, \infty)$ we have $|f(t)| \leq C\exp(at)$. Hence, $g(t) := f(t)\exp(-at)$ is bounded by C, so $g(0^+) = \lim_{s\to\infty} s\mathcal{L}\{g\}(s) = \lim_{s\to\infty} s\mathcal{L}\{f\exp(-at)\}(s) = \lim_{s\to\infty}(s-a)\mathcal{L}\{f\}(s) = \lim_{s\to\infty} s\mathcal{L}\{f\}(s)$.

□

Example. Let $I := \int_0^\infty \frac{\sin(t)}{t}\,dt$. We will compute this value in two different approaches. One is a cumbersome approach that involves an integration technique, the other is applying the theory we have just seen.
If we let $I(a) := \int_0^\infty \exp(-at)\frac{\sin(t)}{t}\,dt$, then $I = I(0)$. Observe that $I'(a) = -\int_0^\infty \exp(-at)\sin(t)\,dt$. We can now apply integration by parts:

$$\begin{aligned} I'(a) &= -\int_0^\infty \exp(-at)\sin(t)\,dt \\ &= \frac{\exp(-at)}{a}\sin(t)\Big|_0^\infty - \frac{1}{a}\int_0^\infty \exp(-at)\cos(t)\,dt \\ &= -\frac{1}{a}\Big(-\frac{\exp(-at)}{a}\cos(t)\Big|_0^\infty + \frac{1}{a}\int_0^\infty \exp(-at)\sin(t)\,dt\Big) \\ &= -\frac{1}{a^2}(1 - I'(a)) \end{aligned}$$

Hence $I'(a) = -\frac{1}{1+a^2}$. Consider that $\int_\infty^a I'(t)\,dt = I(a)$. So finally, we are looking at

$$I = I(0) = \int_0^\infty \frac{1}{1+a^2}\,da = \frac{\pi}{2}$$

A quicker argument, however, is that we already know some rules using the Laplace transform. Observe that I as a function of a is exactly the Laplace transform of the original integrand. But we also know that for any differentiable function, we have $\mathcal{L}\{t \mapsto -tf(t)\} = f'$, hence $I'(a) = -\mathcal{L}\{\sin\} = -\frac{1}{1+a^2}$ indeed using integration by parts or just remembering the formula. All of this, however, is under the assumption that the integrand uniformly converges as $a \to 0$. This we can show. Let $a, t > 0$.

$$\begin{aligned} \left|\exp(-at)\frac{\sin(t)}{t} - \frac{\sin(t)}{t}\right| &= \frac{|\sin(t)|}{t}(1 - \exp(-at)) \\ &\leq \frac{1 - \exp(-at)}{t} \end{aligned}$$

Observe that the last expression is continuous and decreasing in t on $(0, \infty)$. Hence its supremum is the limit at 0:

$$\lim_{t \to 0^+} \frac{1 - \exp(-at)}{t} = \lim_{t \to 0^+} a\exp(-at) = a$$

Hence

$$\sup_{t \in (0,\infty)} \left|\exp(-at)\frac{\sin(t)}{t} - \frac{\sin(t)}{t}\right| \leq a \overset{a \to 0^+}{\to} 0$$

Theorem. *Let $f, g : \mathbb{R} \to \mathbb{C}$ be integrable signals on all compact subsets of $[0, \infty)$. Then*

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$$

where $f * g : [0, \infty) \cup \{0^-\} \to \mathbb{C}$ is defined by

$$(f * g)(t) := \int_0^t f(\tau)g(t - \tau)\,d\tau$$

where 'plugging in' 0^- shall be identified by the lower limit, if it exists.