

1 Conjugates

Definition 1.1. Let F be a field. A field automorphism $\bar{\cdot} : F \rightarrow F$ is called a **conjugate** on F if

- $\bar{\cdot}$ is an involution, i.e. $\bar{\bar{\cdot}} = \text{id}$
- The fixed points of $\bar{\cdot}$, that is the set

$$F^{\bar{\cdot}} := \{x \in F \mid \bar{x} = x\}$$

is an **ordered subfield** of F , when inheriting addition and multiplication from F .

- For all $f \in F$, $f\bar{f} \geq 0_F$.

Lemma 1.1. The usual conjugate on \mathbb{C} given by

$$\overline{x + iy} := x - iy$$

is indeed a conjugate on \mathbb{C} .

Proof. It is trivial to show that $\bar{\cdot}$ is a field automorphism. We can show that it is an involution and in particular invertible by (for all $x, y \in \mathbb{R}$)

$$\overline{\overline{x + iy}} = \overline{x - iy} = x + iy$$

therefore, $\bar{\bar{\cdot}} = \text{id}_{\mathbb{C}}$. Now let $z := x + iy \in \mathbb{C}$ be a fixed point of $\bar{\cdot}$. Then

$$x + iy = x - iy \implies y = 0 \implies z = x \in \mathbb{R}$$

which clearly is ordered by inheriting the usual order on \mathbb{R} , and reversing the embedding $\mathbb{R} \rightarrow \mathbb{C}$ (which is invertible onto its image, and we have just shown that fixed points of the conjugate lie in its image). The last property follows from

$$(x + iy)(x - iy) = x^2 + y^2 \geq 0$$

□

Definition 1.2. Let F be a field with conjugate, define the set

$$F_{\geq 0}^{\bar{\cdot}} := \{f \in F^{\bar{\cdot}} \mid f \geq 0\}$$

We also define the following map, called the **induced absolute value**

$$\begin{aligned} |\cdot| : F &\rightarrow F_{\geq 0}^{\bar{\cdot}} \\ f &\mapsto \sqrt{f\bar{f}} \end{aligned}$$

where $\sqrt{\cdot}$ is the inverse of $f \mapsto f \cdot f$ as a map $F_{\geq 0}^{\bar{\cdot}} \rightarrow F_{\geq 0}^{\bar{\cdot}}$, provided this is invertible.

Lemma 1.2. The induced absolute value is an absolute value, in the sense that it is positive definite, multiplicative and satisfies the triangle inequality, if F has infinite characteristic.

Proof. We show the individual properties.

- It suffices to show for positive definiteness that if for $f \in F$ we have $|f| = 0$ then $f = 0$. Well

$$|f| = 0 \implies \sqrt{f\bar{f}} = 0 \implies f\bar{f} = 0 \implies f = 0$$

where the last equality holds because we have that $f = 0$ or $\bar{f} = 0$, which are actually equivalent.

- Let $a, b \in F$, then

$$|ab| = \sqrt{ab\bar{a}\bar{b}} = \sqrt{a\bar{a}b\bar{b}} = \sqrt{a\bar{a}}\sqrt{b\bar{b}} = |a||b|$$

- Let $a, b \in F$, then we firstly compute for all $t \in F^{\bar{\cdot}}$

$$\begin{aligned} 0 \leq |a - tb|^2 &= (a - tb)\overline{a - tb} \\ &= a\bar{a} - 2t\operatorname{In}(a\bar{b}) + t^2b\bar{b} \end{aligned}$$

which shows by polynomial theorems (that just require ordered fields) that

$$\operatorname{In}(a\bar{b}) \leq |a||b|$$

We now return to

$$\begin{aligned} |a + b| &= \sqrt{(a + b)\overline{a + b}} \\ &= \sqrt{a\bar{a} + \bar{b} + b\bar{a} + \bar{b}} \\ &= \sqrt{a\bar{a} + b\bar{b} + a\bar{b} + b\bar{a}} \\ &= \sqrt{a\bar{a} + b\bar{b} + a\bar{b} + \overline{a\bar{b}}} \\ &= \sqrt{a\bar{a} + b\bar{b} + 2\operatorname{In}(a\bar{b})} \\ &\leq \sqrt{a\bar{a} + b\bar{b} + 2\sqrt{a\bar{a}}\sqrt{b\bar{b}}} \\ &= \sqrt{(\sqrt{a\bar{a}} + \sqrt{b\bar{b}})^2} \\ &= |a| + |b| \end{aligned} \quad \square$$

Definition 1.3. Let $f \in F$, where F is a field with conjugate with $\operatorname{char}(F) \neq 2$ (in the rest of the notes we will further assume this). We define the **invariant** (under conjugate) part of f , $\operatorname{In}(f)$, by

$$\operatorname{In}(f) := \frac{f + \bar{f}}{2}$$

Lemma 1.3. *The invariant part is a fixed point of the conjugate.*

Proof. By direct computation for all $f \in F$,

$$\overline{\left(\frac{f + \bar{f}}{2}\right)} = \frac{1}{2}\overline{f + \bar{f}} = \frac{1}{2}(\bar{f} + f) = \frac{f + \bar{f}}{2}$$

which shows $\overline{\operatorname{In}(f)} = \operatorname{In}(f)$. □

2 Inner products

Definition 2.1. Let V be an F -vector space, where F is a field with a conjugate $\bar{\cdot}$. A map $\langle \cdot, \cdot \rangle : V \times V \rightarrow F$ is called an **inner product** if

- For all $x, y \in V$ we have $\langle x, y \rangle = \overline{\langle y, x \rangle}$
- The map $x \mapsto \langle x, y \rangle$ is linear for all $y \in V$
- For all $x \in V \setminus \{0_V\}$, $\langle x, x \rangle > 0$

Remark 2.1. The last condition is well-defined because

$$\langle x, x \rangle = \overline{\langle x, x \rangle}$$

which makes $\langle x, x \rangle$ be a fixed point of the conjugate, thereby being a member of the ordered subfield $F^\bar{\cdot}$.

Lemma 2.1. Usual inner products over \mathbb{C} -vector spaces are also inner products in this generalized sense.

Theorem 2.1. Let V be an F -vector space with inner product.

- For all $x \in V$, $\langle 0_V, x \rangle = \langle x, 0_V \rangle = 0_F$
- For all $x \in V$, $\langle x, x \rangle \geq 0$
- For all $x, y \in V$ and $a \in F$, $\langle x, ay \rangle = \bar{a}\langle x, y \rangle$
- For all $x, y \in V$,

$$\langle x + y, x + y \rangle = \langle x, x \rangle + 2\operatorname{In}(\langle x, y \rangle) + \langle y, y \rangle$$

Proof. All of them are easy exercises to show except perhaps the first and second to last property. Note that

$$\langle 0_V, x \rangle = \langle 0_F 0_V, x \rangle = 0_F \langle 0_V, x \rangle = 0_F = 0_F \langle x, 0_V \rangle = \langle x, \overline{0_F} 0_V \rangle = \langle x, 0_V \rangle$$

where $\overline{0_F} = 0_F$ due to $\bar{\cdot}$ being a field homo.

We also have

$$\langle x, ay \rangle = \overline{\langle ay, x \rangle} = \overline{a\langle y, x \rangle} = \bar{a}\overline{\langle y, x \rangle} = \bar{a}\langle x, y \rangle \quad \square$$

Definition 2.2. Let $(V, +_V, \cdot_V)$ be an F vector space where F is a field with conjugate. We define the set $\bar{V} := V$, $+_{\bar{V}} := +_V$ and

$$\begin{aligned} \cdot_{\bar{V}} : F \times \bar{V} &\rightarrow \bar{V} \\ \lambda, v &\mapsto \bar{\lambda}v \end{aligned}$$

Then $(\bar{V}, +_{\bar{V}}, \cdot_{\bar{V}})$ is an F vector space, called the **conjugate vector space** of V .

Lemma 2.2. An inner product $\langle \cdot, \cdot \rangle : V \times V \rightarrow F$ is a bilinear map when we view it as a map $V \times \bar{V} \rightarrow F$.

Corollary. An equivalent definition of an inner product is a linear map $V \times \bar{V} \rightarrow F$ that is symmetric and (in some sense) positive definite.

Theorem 2.2. Let V be an inner product space. We define the **induced norm**

$$\begin{aligned} \|\cdot\| : V &\rightarrow F^\bar{\cdot} \\ v &\mapsto \sqrt{\langle v, v \rangle} \end{aligned}$$

where $\sqrt{\cdot}$ is the inverse of $f \mapsto f \cdot f$ as a map $F_{\geq 0}^\bar{\cdot} \rightarrow F_{\geq 0}^\bar{\cdot}$, provided this is invertible. Then V equipped with this induced norm is a normed vector space.

Proof. We show every property of a norm:

- Let $v \in V$ and suppose $\|v\| = 0$, then $\langle v, v \rangle = 0$ which gives $v = 0$

- Let $\lambda \in F$, $v \in V$, then

$$\|\lambda v\| = \sqrt{\langle \lambda v, \lambda v \rangle} = \sqrt{|\lambda|^2 \langle v, v \rangle} = |\lambda| \sqrt{\langle v, v \rangle} = |\lambda| \|v\|$$

- Let $v, w \in V$, then note that

$$\operatorname{In}(\langle v, w \rangle) \leq \|v\| \|w\|$$

which has a similarly spirited proof as before, so that

$$\begin{aligned} \|v + w\| &= \sqrt{\langle v + w, v + w \rangle} \\ &= \sqrt{\langle v, v \rangle + 2 \operatorname{In}(\langle v, w \rangle) + \langle w, w \rangle} \\ &\leq \sqrt{\left(\sqrt{\langle v, v \rangle} + \sqrt{\langle w, w \rangle}\right)^2} \\ &= \|v\| + \|w\| \end{aligned}$$

□

Theorem 2.3 (Cauchy-Schwarz). *For all $v, w \in V$, we have*

$$|\langle v, w \rangle| \leq \|v\| \|w\|$$

Proof. Wlog suppose $w \neq 0_V$, otherwise the inequality is trivial. Then

$$|\langle v, w \rangle| = \frac{\langle v, w \rangle^2}{|\langle v, w \rangle|} = \left\langle v, \frac{\overline{\langle v, w \rangle}}{|\langle v, w \rangle|} w \right\rangle = \operatorname{In} \left(\left\langle v, \frac{\overline{\langle v, w \rangle}}{|\langle v, w \rangle|} w \right\rangle \right) \leq \|v\| \|w\|$$

due to the inequality we proved before, that

$$\operatorname{In}(\langle v, w \rangle) \leq \|v\| \|w\|$$

□