Kenmerk: EWI2019/TW/DMMP/MU/

Test Exam 1, Module 7
Discrete Structures & Efficient Algorithms

All answers need to be motivated. No calculators. You are allowed to use a handwritten cheat sheet (A4, both sides) per topic (ADS, DM).

This exam consists of two parts, with the following (estimated) times per part:

Algorithms & Data Structures (ADS)	1h	(30 points)
Discrete Mathematics (DM)	2h	(60 points)

Total of 30+60=90 points. Your exam grade is the maximum of 1 and the total number of points divided by 9, rounded to one digit.

Please use a new sheet of paper for each part (ADS, DM)!

Algorithms & Data Structures

1. (10 points) Consider this sorting algorithm that sorts from a sequence A of integers the segment $A[i, \ldots, j+1]$ where $1 \leq i \leq j$:

```
def sort(A,i,j):
    if A[i]>A[j] : A[i],A[j]=A[j],A[i]
    if i+1>=j : return
    k=(j-i+1)//3
    sort(A,i,j-k)
    sort(A,i+k,j)
    sort(A,i,j-k)
```

   (a) Determine the asymptotic worst-case complexity for *sort* to sort $n > 0$ numbers. Take as basic operation the comparison of elements of A.

   (b) Under which circumstances would you prefer *sort* over quicksort, insertion sort, mergesort or heapsort?

2. (10 points) Given a binary search tree where all keys are unique positive numbers. Give an algorithm that for a node with key k yields the node with the biggest value smaller than k (and *nil* if there is no such node).

3. (10 points) Consider the following game. You have a board with n by n squares. You can put a checkers stone on a square on the lower row of the board. Then you can push the stone diagonally upwards, as long as you stay on the board. For each move you get a (positive) number of points, given by a table: from (i, j) right upwards yields $p(i, j, R)$ points, left upwards yields $p(i, j, L)$ points. You finally end in a square on the upper row. Suppose the coordinates of the lowest leftmost square are $(1, 1)$.

   Suppose the maximal number of points when arriving in square (i, j) is $R(i, j)$. Suppose that $1 \leq j \leq n$ and $0 \leq i \leq n+1$, with $R(0, j) = 0$ and $R(n+1, j) = 0$, and all moves from and to squares with $i = 0$ or $i = n+1$ yield 0 points. Then the following recurrent equation holds:

   - $R(i, j) =$
     $max\{R(i-1, j-1) + p(i-1, j-1, R), R(i+1, j-1) + p(i+1, j-1, L)\}$
     for $1 \leq i \leq n$, $1 < j \leq n$
   - $R(i, j) = 0$ for $i = 0$ or $i = n+1$ or $j = 1$

   Give and algorithm to determine the maximal number of points you can win. The complexity of the algorithm may be no worse than quadratic in n.

Discrete Mathematics

4. (10 points)

   (a) Show that the Diophantine equation $1000s + 444t = 2$ has no solution for $s, t \in \mathbb{Z}$.

   (b) Show that for $a, b, c \in \mathbb{Z}$, if gcd$(a, b)|c$, then equation $as + bt = c$ has a solution in integers (s, t).

5. (10 points)

   (a) Compute the solution to the recurrence relation

   $$a_n - 10a_{n-1} + 21a_{n-2} = 60 \cdot 3^n \quad (n \geq 2) \quad \text{with} \quad a_0 = 2 \text{ en } a_1 = -5\,.$$

   (b) Consider strings in $\{0, 1, 2\}^*$. Let a_n be the number of strings in $\{0, 1, 2\}^*$ of length n that do not contain the substring 01 and neither 02. Compute a_1, a_2, a_3 and a recurrence relation for a_n for all $n \geq 4$. (You do not need to solve this recurrence relation.)

6. (10 points) Let $G = (V, E)$ be a simple, undirected graph, with edge weights $d_e \geq 0$, $e \in E$. Let $T \subseteq E$ be a minimum weight spanning tree (MST) for G. Moreover, for a fixed node $s \in V$, let us denote by $D(s, v)$ all edges that lie on shortest (s, v)-paths, for $v \in V$. Let $D(s) := \bigcup_{v \in V} D(s, v)$. Prove that $T \cap D(s) \neq \emptyset$.

7. (10 points) Let $G = (V, E)$ be a simple, bipartite undirected graph. Let $|V| = n$ and $|E| = m > 1$. Prove or give a counterexample:

   (a) If $m \leq 2n - 4$, Then G is planar.

   (b) If G is planar, then $m \leq 2n - 4$.

8. (10 points) How many possibilities are there to distribute 50€ over three persons, such that nobody gets less than 10€, and somebody gets at most 15€? Use a generating function to compute the answer.

9. (10 points) Give a short proof or give a counterexample for each of the following statements.

   (a) Consider an undirected, simple graph $G = (V, E)$ with edge weights $w_e \geq 0$, $e \in E$. Then any two minimum spanning trees T_1 and T_2 for G must have a nonempty intersection, that is, $T_1 \cap T_2 \neq \emptyset$.

   (b) Consider a capacitated network $G = (V, A, c)$, where V is the set of vertices, A is the set of directed arcs, and $c_a \geq 0$, $a \in A$ are the arc capacities. Then there always exists a maximum flow f_a, $a \in A$, such that either $f_a = 0$ or $f_a = c_a$ for all $a \in A$.

   (c) Consider an undirected, simple graph $G = (V, E)$ with edge weights $w_e \geq 0$ such that $w_e \neq w_{e'}$ for all $e, e' \in E$, $e \neq e'$. Let $s \in V$ be fixed. Then for all $v \in V$ there is a unique shortest (s, v)-path.

   (d) Consider an undirected, simple graph $G = (V, E)$ with edge weights $w_e \geq 0$ such that $w_e \neq w_{e'}$ for all $e, e' \in E$, $e \neq e'$. Then there is a unique minimum spanning tree T.