4. (a) V has four elements, S_3 has six elements, V can only be isomorphic to a subgroup of S_n if four divides the number of elements in S_n.

   (b) All nonzero elements of V have order 2. Take $H = \{\epsilon, (12), (34), (12)(34)\}$. Corresponding isomorphism: $\phi((0,0)) = \epsilon$, $\phi((1,0)) = (12)$, $\phi((0,1)) = (34)$, $\phi((1,1)) = (12)(34)$.

5. (a) $Z(G)$ is a subgroup of G if for all $h_1, h_2 \in Z(G)$ also $h_1h_2 \in Z(G)$ and $h \in Z(G)$ implies that $h^{-1} \in Z(G)$. If $h_1, h_2 \in Z(G)$ then for all $g \in G$:

$$h_1h_2g = h_1gh_2 = gh_1h_2 \quad hg = gh \Rightarrow g = h^{-1}gh \Rightarrow gh^{-1} = h^{-1}g.$$

   (b) Take:

$$g_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

   The only matrices that commute with g_1 are diagonal matrices:

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

   Now take:

$$g_2 = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

   The only diagonal matrices that commute with g_2 are diagonal matrices of the form:

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

   (c) Define $\phi : Z(G) \to \mathbb{R}$ by

$$\phi(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}) = a$$

   Check that this defines an ismorphism.

6. (a) $p(a) \neq 0$ for all $a \in \mathbb{Z}_5$, and $\deg p(x) = 3$, so $p(x)$ is irreducible.

   (b) The ideal I is generated by an irreducible polynomial and is therefore maximal in $\mathbb{Z}_5[x]$, therefore $\mathbb{Z}_5[x]/I$ is a field.

   (c) All elements of $\mathbb{F}$ are of the form

$$a_0 + a_1x + a_2x^2 + I \quad a_i \in \mathbb{Z}_5.$$

   It follows that $\mathbb{F}$ has 5^3 elements.

   (d) Long division of $p(x)$ by $2x + 3$ yields:

$$p(x) = (3x^2 + 4x + 4)(2x + 3) + 4.$$

   And therefore the inverse is given by:

$$3x^2 + 4x + 4 + I.$$

   (e) $x^3 + 3x + 2$ is also irreducible and of degree 3 and hence $\mathbb{Z}_5[x]/ < x^3 + 3x + 2 >$ is a field of 5^3 elements. Since fields of the same number of elements are isomorphic, the statement follows.

7. For each edge of the square we can choose from five colors. This yields that there are 5^4, i,e, 625 ways to paint the square. However, there are four rotations and four reflections that reduce the number of different possibilities.

   Since the square is made of iron wire, we can take D_4 as the group of symmetries. Then:

$$|\text{Fix}(R_0)| = 625 \quad |\text{Fix}(R_{90})| = 5 \quad |\text{Fix}(R_{180})| = 25 \quad |\text{Fix}(R_{270})| = 5$$

   and

$$|\text{Fix}(H)| = 5^3 = 125 \quad |\text{Fix}(V)| = 5^3 = 125 \quad |\text{Fix}(D)| = 5^2 = 25 \quad |\text{Fix}(D')| = 5^2 = 25$$

   Using Burnside's theorem it follows that the number of orbits, that is the number of different colorings is:

$$\frac{1}{8}(625 + 5 + 25 + 5 + 125 + 125 + 25 + 25) = 120.$$

8.

$$\alpha = (15)(234) = (15)(24)(23).$$