Languages and Machines (Module 7 TCS+IAM)

L&M 8: Turing Machines and Unrestricted Grammars

(Undecidability of the Halting Problem)

Ch8:1-7; Ch11:1; Ch11:4-5; Ch12:1; Ch12:4

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Lecture 8

Recall Turing Machines

Turing Machines:

- Finite Automaton + Unbounded Tape
  - Read/Modify the tape and move to the Left/Right
  - $\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$

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- Equivalent Alternatives
  - Standard Turing Machine (1 tape, deterministic)
  - 2-sided tape, multiple tracks, multiple tapes
  - Non-deterministic Turing Machine (backtracking)

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  - Standard Turing Machine (1 tape, deterministic)
  - 2-sided tape, multiple tracks, multiple tapes
  - Non-deterministic Turing Machine (backtracking)
- Language Acceptance by TM
  - TM accepts w if TM halts in a final state
  - L is recursively enumerable: L is recognised by some TM, which then accepts precisely all words $w \in L$.
  - L is recursive: L is decided by some TM, which accepts all words $w \in L$, and moreover it *halts on all inputs in* Σ^* and

Contents

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Unrestricted grammars

Left side contains (besides a variable) *context information*

Grammar:

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\begin{aligned}
S &\rightarrow aAbc \mid \lambda \\
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Cb &\rightarrow bC \\
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Definition

An *unrestricted grammar* is a tuple $\langle V, \Sigma, P, S \rangle$ with

- V a set of variables
- Σ the alphabet
- $P \subseteq (V \cup \Sigma)^+ \times (V \cup \Sigma)^*$ a set of rules
- $S \in V$ the start symbol.

Example: unrestricted grammar for $w[w]$

The following grammar generates the language $\{w[w] \mid w \in \{a,b\}^*\}$:

$$S \to aT[a] \mid bT[b] \mid []$$
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Functioning (W represents w in uppercase):

- S and T generate $\{xw[W^R x] \mid x \in \{a,b\}, w \in \{a,b\}^*\}$
- A and B then propagate to the right; thus $[W^R x]$ turns into $[xw]$

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Relation unrestricted grammars $\leftrightarrow$ Turing machines

Theorem (Theorem 10.1.2 and 10.1.3)

L recursively enumerable $\Leftrightarrow$ $\exists$ unrestricted grammar G with $L = L(G)$

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Construct a Turing machine M_G such that $L(M_G) = L(G)$

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"Only If" ($\Rightarrow$)

Given a Turing machine M, construct an unrestricted grammar G_M such that $L(G_M) = L(M)$.

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2. Encode the rules of G on Tape 2; for example

   $S \to Abc \mid \lambda \qquad A \to aAbC \mid \lambda \qquad Cb \to bC \qquad Cc \to cc$

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Terminate if Tapes 1 and 3 are equal

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Idea of the functioning of G_M in three phases:

- Rules to generate any word $w[q_0 Bw]$ from S
  - The first part w is the word that will be recognized
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  - For all $q_i \xrightarrow{x/z\ R} q_j$ and all $y \in \Gamma$, we include a rule $q_ixy \to zq_jy$ in the grammar

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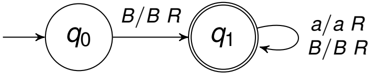
- Remove the final part $[\cdots]$ only if a final state is reached
  - This can be done with a number of simple "Erasure" rules

Example: Turing machine to unrestricted grammar

- Turing machine for the (regular) language $a^* b(a \cup b)^*$:



- Symbols of G_M:
  $\Sigma = \{a, b\}, V = \{S, T, E_R, E_L, [,], A, X, B, q_0, q_1\}$

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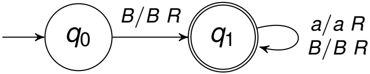

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Phase 1 ($z \in \{a, b\}$)

$$S \to zT[z] \mid [q_0B]$$
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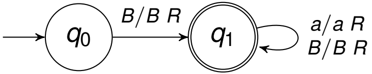

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Phase 2 ($z \in \{a, b, B\}$)

$$q_0 B z \to B q_1 z$$
$$q_0 B] \to B q_1 B]$$
$$q_1 a z \to a q_1 z$$
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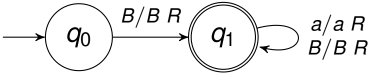

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$S \to zT[z] \mid [q_0B]$	$q_0Bz \to Bq_1z$	$q_1b \to E_R$
$T[\to aT[X \mid bT[Y \mid [q_0B$	$q_0B] \to Bq_1B]$	$E_Rz \to E_R$
$Xz \to zX$	$q_1az \to aq_1z$	$E_R] \to E_L$
$Yz \to zY$	$q_1a] \to aq_1B]$	$zE_L \to E_L$
$X] \to a]$	$q_1Bz \to Bq_1z$	$[E_L \to \lambda$
$Y] \to b]$	$q_1B] \to Bq_1B]$	

- Example: derivation of *aaba*

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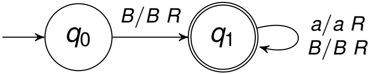

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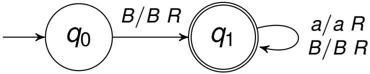

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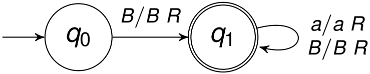

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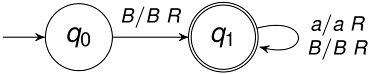

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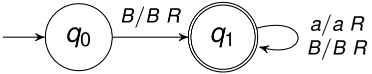

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Classification of languages (sets of words over Σ^*):

Chomsky Hierarchy:

Type	Languages	Grammars	Machines
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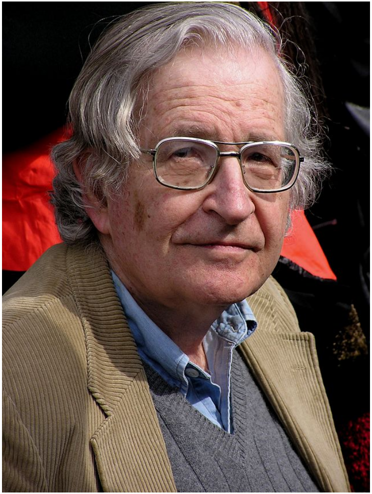
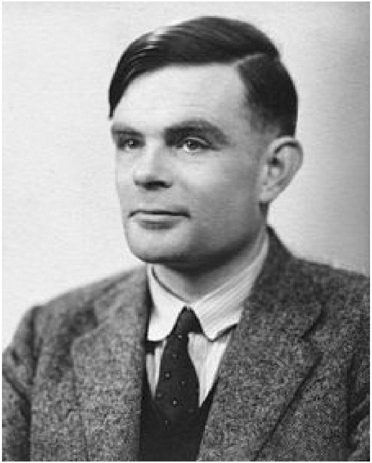
Heroes and Pioneers



Noam Chomsky (1928)



Alan Turing (1912-1954)

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What should be the input to that Universal Turing Machine?

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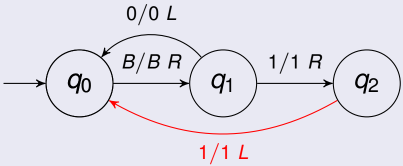

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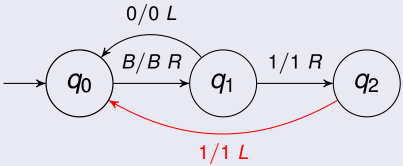

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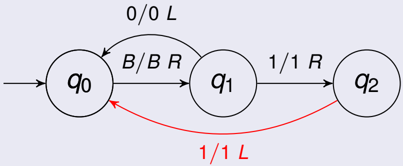

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The Universal Turing Machine

We can now build one Universal Turing Machine U:

For every Turing Machine M the following applies:

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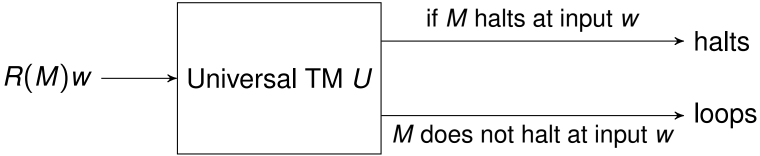

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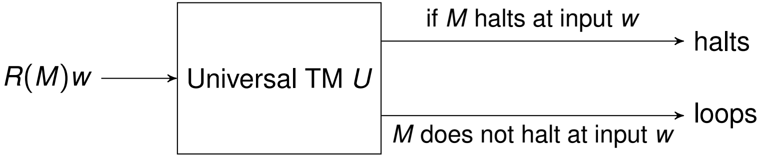

So $\mathcal{L} = \{R(M)w \mid \text{TM } M \text{ halts with input } w\}$ is recursively enumerable.

Church-Turing thesis

Computer = Universal Turing Machine

U = hardware, $R(M)$ = software (!)

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Every effective procedure can be encoded as a Turing Machine.

But then it becomes interesting to investigate which problems a Turing machine can not solve!

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Decision problems and solutions

Definition: Decision problem

A *decision problem* is a question that takes some input and provides a yes/no answer

Example:

- Is a given number a square?
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Church' thesis: A problem is decidable $\iff$ a TM exists for it

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Using a suitable representation, we have encoded decision problems as input to a Turing Machine.

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- Question: Do undecidable problems exist?
- Equivalent question: Do non-recursive languages exist?
- Answer *yes!* And this puts a limit to what computers can do

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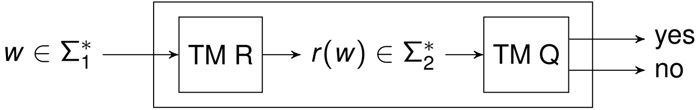

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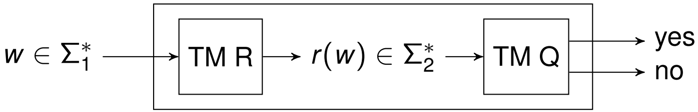

Problem Reduction and Decidability

- If L is reducible to Q and Q is decidable, then L is decidable
- If L is reducible to Q and L is not decidable, then Q is also not decidable

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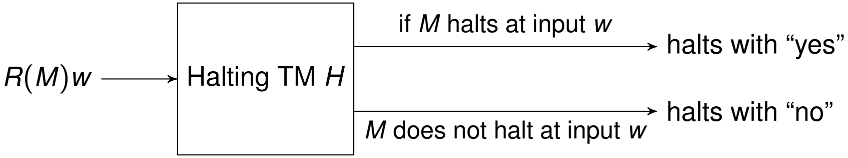

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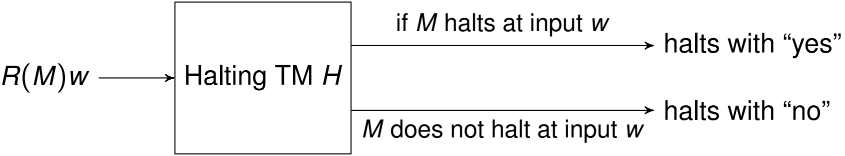


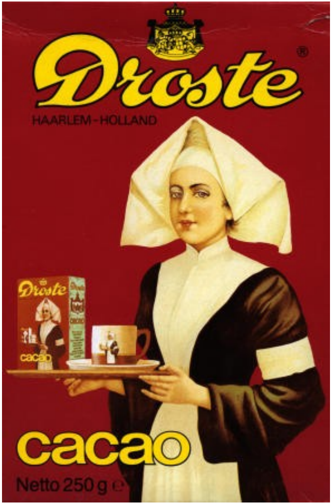
- Turing machine and input word are parameters of this problem
- Required: encoding of Turing machine M: $R(M) \in \Sigma^*$

Theorem

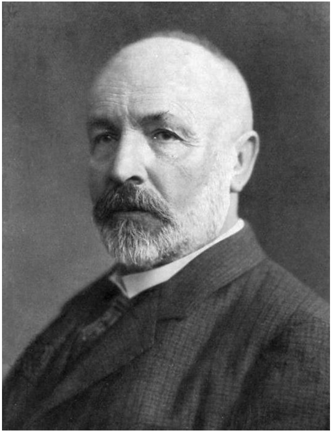
The halting problem is undecidable.

Proof: by contradiction and self-reference (diagonalisation).

Self reference



The Droste effect



Georg Cantor (1845-1918)

(Remember Cantor's diagonal argument?)

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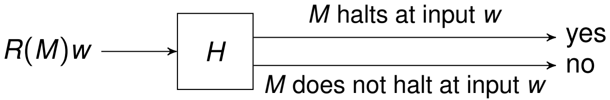

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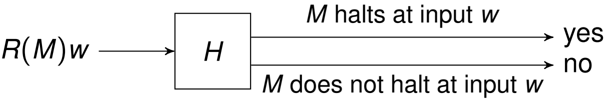

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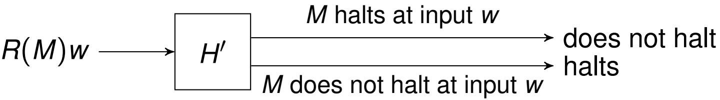

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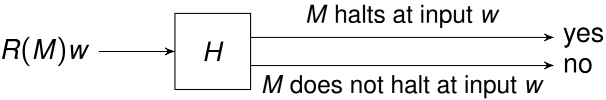

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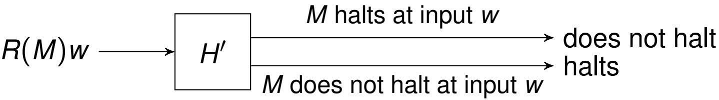

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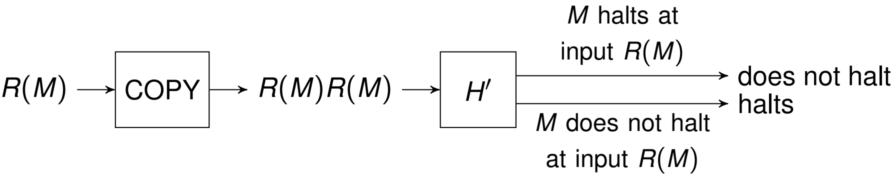

Undecidability of the *halting* problem, continuation

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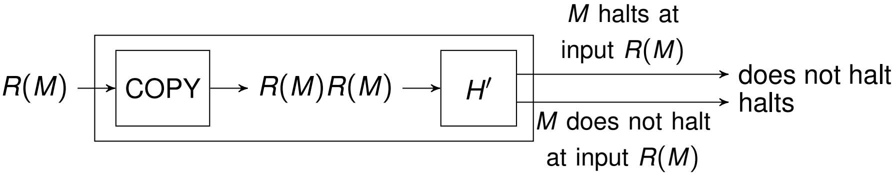

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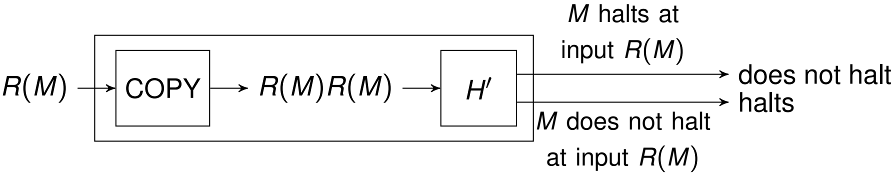
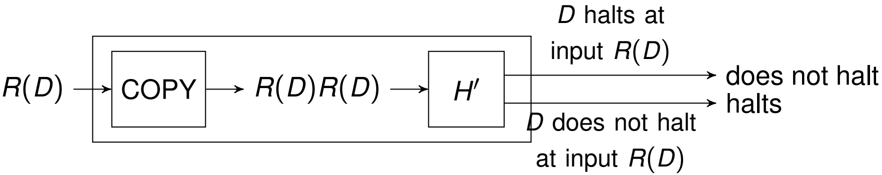
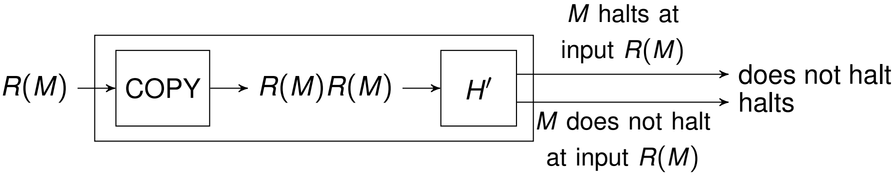


4. Give the encoding of D to itself as input



Undecidability of the *halting* problem, continuation

3. Name the new machine D



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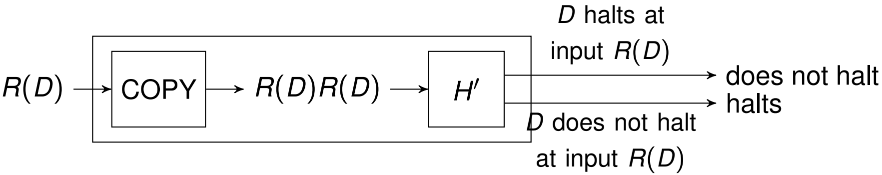

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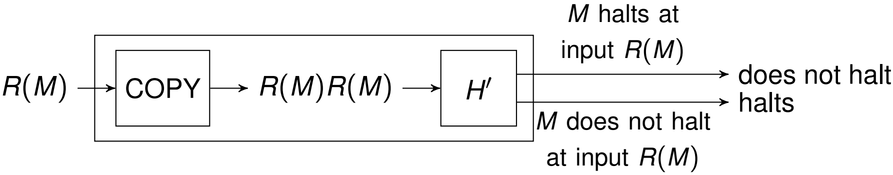

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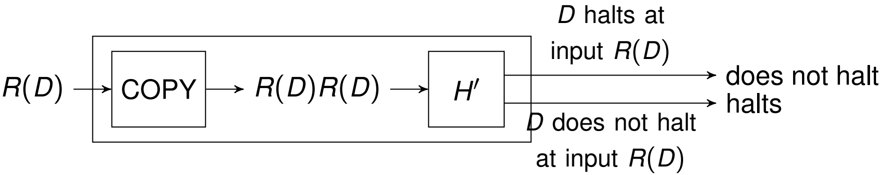

5. Contradiction: the computation halts $\iff$ the computation does not terminate
6. Conclusion: the original machine H cannot exist

Direct consequences of the Halting Problem

- There is no program that can check the termination of all other programs, which (given unbounded resources) always terminates with the correct answer.
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- Example:
  - The language of all syntactically correct Python programs is recursive
  - The language of all *terminating* Pyton programs is recursively enumerable, but not recursive!

More Consequences

By Problem Reduction: many other problems are also undecidable

- Blank Tape Problem – does M halt with an empty tape as input?
- Strategy: Reduce Halting Problem to Blank Tape Problem (!)
- Input for Halting Problem: $R(M)w$.
- Transform this input to: $R(M')$, where M' first writes word w and then executes M on w.
- Blank Tape for $R(M') \iff$ Halting for $R(M)w$.

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Rice's Theorem

Every non-trivial property of recursively enumerable languages of the form $\mathcal{L}(M)$ is undecidable

Examples:

- Is $\mathcal{L}(M)$ empty? context-free? regular? Is $\lambda \in \mathcal{L}(M)$?
- Are $\mathcal{L}(M)$ and $\mathcal{L}(M')$ equivalent?

Other Undecidable Problems

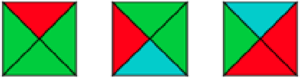
Decidable:

- Given CFG G and word w: is $w \in \mathcal{L}(G)$?

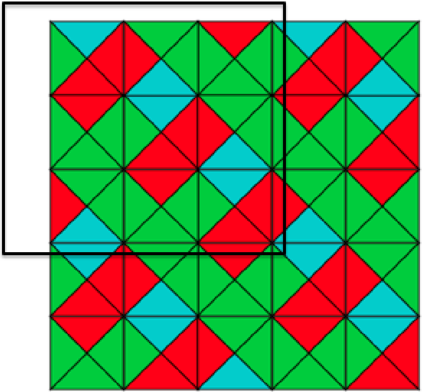
Undecidable:

- Given CFG G_1 and G_2. Is $\mathcal{L}(G_1) = \mathcal{L}(G_2)$?
- Post's Correspondence Problem:
  - Input: two finite sequences of words ("dominos") $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$.
  - Question: is there a finite sequence of dominos (with copies) such that $x_{f(1)}, \ldots, x_{f(m)} = y_{f(1)}, \ldots y_{f(m)}$?
- Tiling Problem: Given n tiles, can we tile any m x m area?

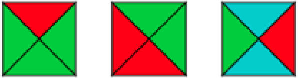
Positive Tiling Instance



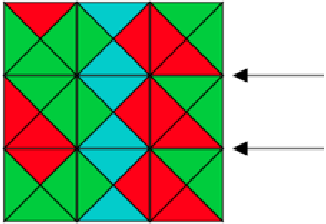
YES



Negative Tiling Instance



NO



Seen this week

Chomsky Hierarchy

- Turing Machines
  - Multi-tape, Non-deterministic – it does not matter
- Unrestricted grammars
  - Equivalent to Turing machines (recursively enumerable languages)
- Seen before: Context-free grammars
  - Equivalent to pushdown automata
- Regular grammars
  - Equivalent to finite automata and regular expressions

Undecidability

- Not all problems can be solved efficiently (P vs NP)
- Some problems cannot be solved at all! (halting problem)

Watch this great animation:

https://www.youtube.com/watch?v=92WHN-pAFCs