Languages and Machines (Module 7 TCS+IAM)

L&M 8: Turing Machines and Unrestricted Grammars

(Undecidability of the Halting Problem)

Ch8:1-7; Ch11:1; Ch11:4-5; Ch12:1; Ch12:4

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Lecture 8

Recall Turing Machines

Turing Machines:

- Finite Automaton + Unbounded Tape
  - Read/Modify the tape and move to the Left/Right
  - $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$
- Equivalent Alternatives
  - Standard Turing Machine (1 tape, deterministic)
  - 2-sided tape, multiple tracks, multiple tapes
  - Non-deterministic Turing Machine (backtracking)
- Language Acceptance by TM
  - TM accepts w if TM halts in a final state
  - L is recursively enumerable: L is recognised by some TM, which then accepts precisely all words $w \in L$.
  - L is recursive: L is decided by some TM, which accepts all words $w \in L$, and moreover it *halts on all inputs in* Σ^* and

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Unrestricted grammars

Left side contains (besides a variable) *context information*

Grammar:

$$\begin{array}{rcl} S & \to & aAbc \mid \lambda \\ A & \to & aAbC \mid \lambda \\ Cb & \to & bC \\ Cc & \to & cc \end{array}$$

Example derivation:

$$\begin{array}{rl} S & \Rightarrow aAbc \Rightarrow aaAbCbc \\ & \Rightarrow aabCbc \Rightarrow aabbCc \\ & \Rightarrow aabbcc \end{array}$$

Language: $\{a^i b^i c^i \mid i \geq 0\}$ (not context-free!)

Definition

An *unrestricted grammar* is a tuple $\langle V, \Sigma, P, S \rangle$ with

- V a set of variables
- Σ the alphabet
- $P \subseteq (V \cup \Sigma)^+ \times (V \cup \Sigma)^*$ a set of rules
- $S \in V$ the start symbol.

Example: unrestricted grammar for $w[w]$

The following grammar generates the language $\{w[w] \mid w \in \{a,b\}^*\}$:

$$\begin{array}{lll} S \to aT[a] \mid bT[b] \mid [] & Ax \to xA & (x \in \{a,b\}) \\ T[\to aT[A \mid bT[B \mid [& Bx \to xB & (x \in \{a,b\}) \\ & A] \to a] & \\ & B] \to b] & \end{array}$$

Functioning (W represents w in uppercase):

- S and T generate $\{xw[W^R x] \mid x \in \{a,b\}, w \in \{a,b\}^*\}$
- A and B then propagate to the right; thus $[W^R x]$ turns into $[xw]$

Example: derivation of $aab[aab]$

$$\begin{array}{rl} S & \Rightarrow aT[a] \\ & \Rightarrow aaT[Aa] \\ & \Rightarrow aabT[BAa] \\ & \Rightarrow aab[BAa] \end{array} \qquad \begin{array}{rl} aab[BAa] & \Rightarrow aab[BaA] \\ & \Rightarrow aab[aBA] \\ & \Rightarrow aab[aBa] \\ & \Rightarrow aab[aaB] \\ & \Rightarrow aab[aab] \end{array}$$

Relation unrestricted grammars $\leftrightarrow$ Turing machines

Theorem (Theorem 10.1.2 and 10.1.3)

L recursively enumerable $\Leftrightarrow \exists$ unrestricted grammar G with $L{=}L(G)$

"If" ($\Leftarrow$)

Construct a Turing machine M_G such that $L(M_G) = L(G)$

"Only If" ($\Rightarrow$)

Given a Turing machine M, construct an unrestricted grammar G_M such that $L(G_M) = L(M)$.

From unrestricted grammars to Turing machines

"If" ($\Leftarrow$)

Construct a Turing machine M_G such that $L(M_G) = L(G)$

Construction-idea: Use a 3-tape machine:

1. Write the input on Tape 1
2. Encode the rules of G on Tape 2; for example
   $S \to Abc \mid \lambda \quad A \to aAbC \mid \lambda \quad Cb \to bC \quad Cc \to cc$
   is encoded as
   $BS\#Abc\#\#S\#\#\#A\#aAbC\#\#A\#\#\#Cb\#bC\#\#Cc\#ccB$
3. Write the start symbol S on Tape 3

Iteratively (nondeterministically) apply rules $u \to v$ of Tape 2
on a (nondeterministic) partial word u on Tape 3
Terminate if Tapes 1 and 3 are equal

From Turing machines to unrestricted grammars

"Only If" ($\Rightarrow$)

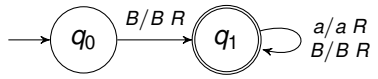
Given a Turing machine M, construct an unrestricted grammar G_M such that $L(G_M) = L(M)$.

Idea of the functioning of G_M in three phases:

- Rules to generate any word $w[q_0Bw]$ from S
  - The first part w is the word that will be recognized
  - The last part $[q_0Bw]$ represents the configuration of TM M
- Give grammar rules that simulate the functioning of M, for example:
  - For all $q_i \xrightarrow{x/z\ R} q_j$ and all $y \in \Gamma$, we include a rule $q_ixy \to zq_jy$ in the grammar
- Remove the final part $[\cdots]$ only if a final state is reached
  - This can be done with a number of simple "Erasure" rules

Example: Turing machine to unrestricted grammar

- Turing machine for the (regular) language $a^*b(a \cup b)^*$:



- Symbols of G_M:
  $\Sigma = \{a, b\}$, $V = \{S, T, E_R, E_L, [,], A, X, B, q_0, q_1\}$

Phase 1 ($z \in \{a, b\}$)

$S \to zT[z] \mid [q_0B]$
$T[\to aT[X \mid bT[Y \mid [q_0B$
$Xz \to zX$
$Yz \to zY$
$X] \to a]$
$Y] \to b]$

Phase 2 ($z \in \{a, b, B\}$)

$q_0Bz \to Bq_1z$
$q_0B] \to Bq_1B]$
$q_1az \to aq_1z$
$q_1a] \to aq_1B]$
$q_1Bz \to Bq_1z$
$q_1B] \to Bq_1B]$

Phase 3

$q_1b \to E_R$
$E_Rz \to E_R$
$E_R] \to E_L$
$zE_L \to E_L$
$[E_L \to \lambda$

- Example: derivation of *aaba*

$$S \Rightarrow^* aaba[q_0Baaba] \Rightarrow^* aaba[Baaq_1ba]$$
$$\Rightarrow aaba[BaaE_Ra] \Rightarrow aaba[BaaE_L \Rightarrow^* aaba$$

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The Chomsky Hierarchie

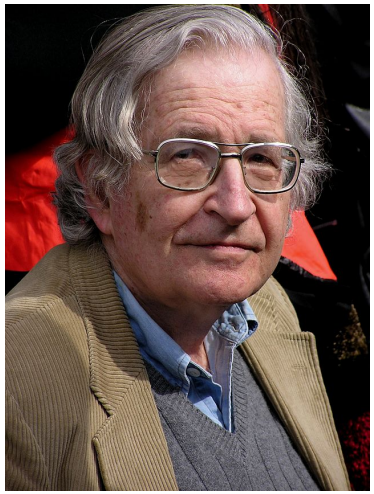
Classification of languages (sets of words over Σ^*):

Chomsky Hierarchy:

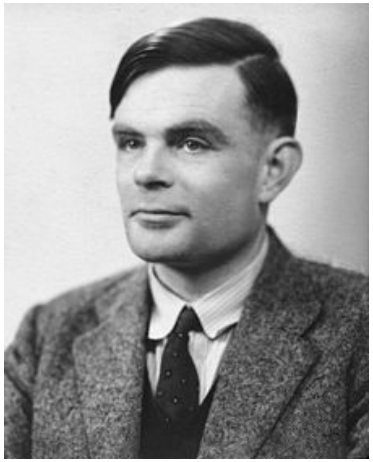
Type	Languages	Grammars	Machines
Type 0	Rec. enumerable	Unrestricted	Turing Machine
	Recursive		(terminates)
Type 2	Context-free	Context-free	Pushdown automaton (PDA)
			Deterministic
Type 3	Regular	Regular	Finite (DFA/NDA)

- Vertically downwards:
  - formalisms become less powerful, languages become smaller
  - easier to analyze
- Horizontal:
  - Various grammars exactly equal to various automata

Heroes and Pioneers



Noam Chomsky (1928)



Alan Turing (1912-1954)

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What is the power of Turing Machines?

We have seen that Turing Machines can simulate:

- Non-deterministic Turing Machines
- Unrestricted Grammars

Challenge

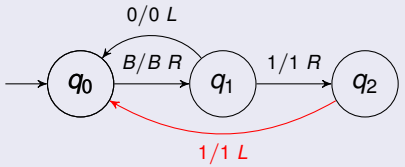
Can we build a single Turing Machine that can simulate all other Turing Machines?

What should be the input to that Universal Turing Machine?

Encode a Turing Machine as input

- A Turing Machine can be encoded in 0's and 1's.
- Let $R(M)$ be the code of Turing Machine M.
- One can then use $R(M)$ as input for another TM M'.

Turing Machine M:



$\delta(q_2, 1) = [q_0, 1, L]$

1110110101101

Encoding:

0	1
1	11
B	111
q_0	1
q_1	11
q_2	111
L	1
R	11

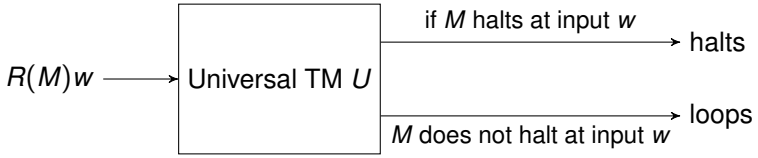
$R(M) = 000101110110111011001101010101001101101110110110011101101011010000$

The Universal Turing Machine

We can now build one Universal Turing Machine U:

For every Turing Machine M the following applies:

- If M halts for input w: Then U also halts for input $R(M)w$
- If M does not terminate for input w: Then U will also not terminate for input $R(M)w$



So $\mathcal{L} = \{R(M)w \mid \text{TM } M \text{ halts with input } w\}$ is recursively enumerable.

Church-Turing thesis

Computer = Universal Turing Machine

U = hardware, $R(M)$ = software (!)

Definition: Effective procedure (= algorithm)

A procedure is effective if it has the following properties:

Complete	The procedure always produces the correct answer
Mechanistic	Carrying out the procedure requires no intelligence
Deterministic	The procedure always returns the same answer

Church-Turing thesis

Every effective procedure can be encoded as a Turing Machine.

But then it becomes interesting to investigate which problems a Turing machine can not solve!

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Decision problems and solutions

Definition: Decision problem

A *decision problem* is a question that takes some input and provides a yes/no answer

Example:

- Is a given number a square?
- Is there a path between node a and b in graph G?

A decision problem can be *encoded* as word on a Turing machine

- For example: by a series of digits, or even a row of ones

Definition: solution to a decision problem by a TM

A Turing machine M solves a decision problem if M always

- Halts in a final state if the answer is “yes”
- Halts in a non-final state if the answer is “no”
- *Always terminates!*

Church’ thesis: A problem is decidable $\iff$ a TM exists for it

Decision Problems as Language Recognition

Using a suitable representation, we have encoded decision problems as input to a Turing Machine.

Decidability of Decision Problems

A (decision) problem is *decidable* iff it is solved by some TM.

So to every decision problem we associate the language
$\mathcal{L}$ = inputs for which the answer to the problem is "yes"

A problem is *decidable* if its corresponding language is *recursive*

- Question: Do undecidable problems exist?
- Equivalent question: Do non-recursive languages exist?
- Answer *yes!* And this puts a limit to what computers can do

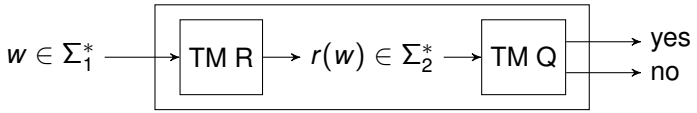
Problem Reduction

A problem can be solved directly by providing a TM that decides it. Alternative, we could reduce it an already solved problem:

Effective Problem Reduction (many-to-one)

$L \subseteq \Sigma_1^*$ is reducible to $Q \subseteq \Sigma_2^*$ if there exists a Turing Machine M that implements a function $r : \Sigma_1^* \to \Sigma_2^*$, such that for all w:

$$w \in L \text{ if and only if } r(w) \in Q$$



Problem Reduction and Decidability

- If L is reducible to Q and Q is decidable, then L is decidable
- If L is reducible to Q and L is not decidable, then Q is also not decidable

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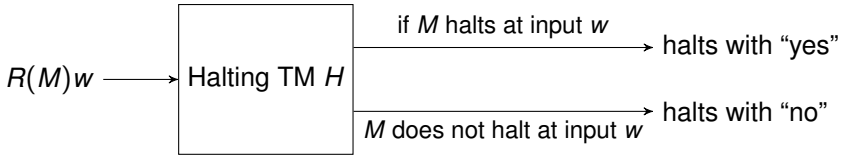
The *Halting* problem

Definition: Halting problem

Will a given Turing machine halt when provided a given input?



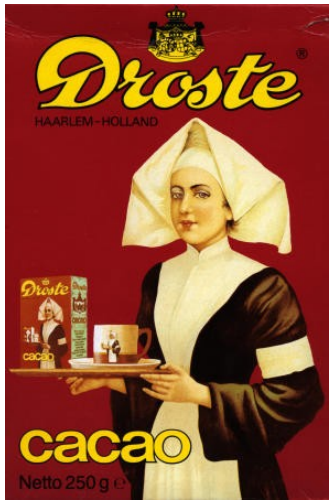
- Turing machine and input word are parameters of this problem
- Required: encoding of Turing machine M: $R(M) \in \Sigma^*$

Theorem

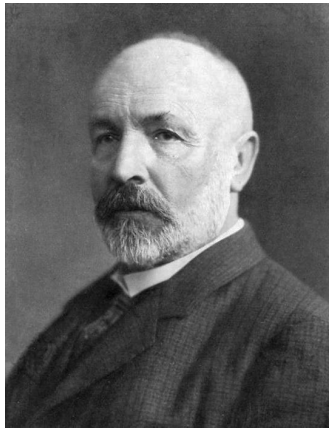
The halting problem is undecidable.

Proof: by contradiction and self-reference (diagonalisation).

Self reference



The Droste effect

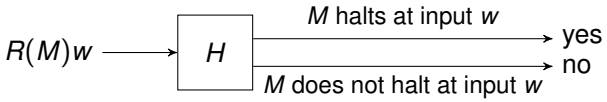


Georg Cantor (1845-1918)

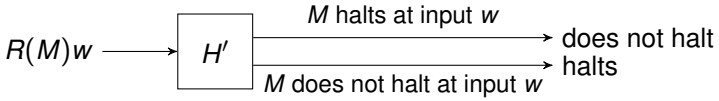
(Remember Cantor's diagonal argument?)

Undecidability of the *halting* problem

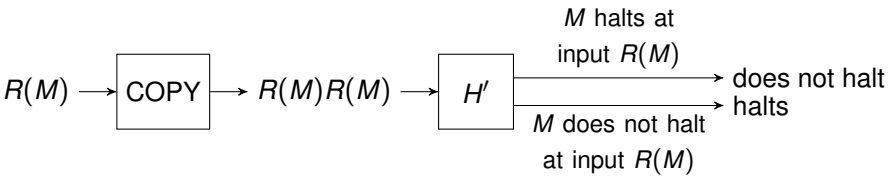
1. Suppose we have a solution H of the halting problem



2. Transform this into H' by not terminating in every successful state

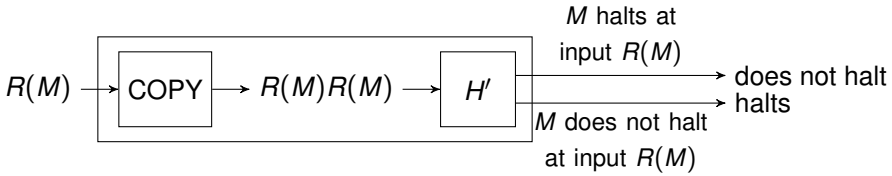


3. Use the encoding $R(M)$ as input w

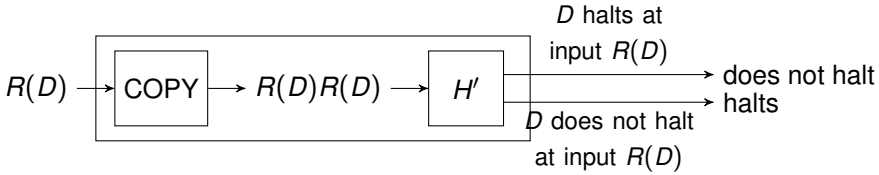


Undecidability of the *halting* problem, continuation

3. Name the new machine D



4. Give the encoding of D to itself as input



5. Contradiction: the computation halts $\iff$ the computation does not terminate
6. Conclusion: the original machine H cannot exist

Direct consequences of the Halting Problem

- There is no program that can check the termination of all other programs, which (given unbounded resources) always terminates with the correct answer.
- This has not stopped scientists to organise termination competitions!
- The class of recursive languages is *strictly smaller* than the class of recursively enumerable languages:
  - The Halting Problem is not a recursive language
  - The Halting Problem is recursively enumerable (universal TM)
- Example:
  - The language of all syntactically correct Python programs is recursive
  - The language of all *terminating* Pyton programs is recursively enumerable, but not recursive!

More Consequences

By Problem Reduction: many other problems are also undecidable

- Blank Tape Problem – does M halt with an empty tape as input?
- Strategy: Reduce Halting Problem to Blank Tape Problem (!)
- Input for Halting Problem: $R(M)w$.
- Transform this input to: $R(M')$, where M' first writes word w and then executes M on w.
- Blank Tape for $R(M') \iff$ Halting for $R(M)w$.

Rice's Theorem

Every non-trivial property of recursively enumerable languages of the form $\mathcal{L}(M)$ is undecidable

Examples:

- Is $\mathcal{L}(M)$ empty? context-free? regular? Is $\lambda \in \mathcal{L}(M)$?
- Are $\mathcal{L}(M)$ and $\mathcal{L}(M')$ equivalent?

Other Undecidable Problems

Decidable:

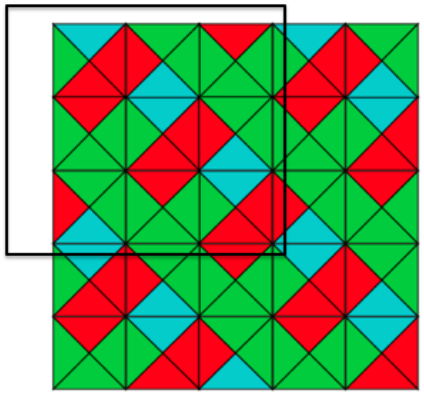
- Given CFG G and word w: is $w \in \mathcal{L}(G)$?

Undecidable:

- Given CFG G_1 and G_2. Is $\mathcal{L}(G_1) = \mathcal{L}(G_2)$?
- Post's Correspondence Problem:
  - Input: two finite sequences of words ("dominos") $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$.
  - Question: is there a finite sequence of dominos (with copies) such that $x_{f(1)}, \ldots, x_{f(m)} = y_{f(1)}, \ldots y_{f(m)}$?
- Tiling Problem: Given n tiles, can we tile any m x m area?

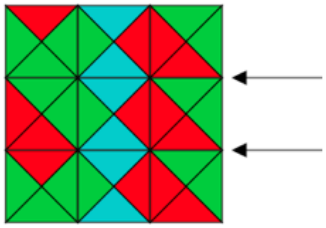
Positive Tiling Instance

YES



Negative Tiling Instance

NO



Seen this week

Chomsky Hierarchy

- Turing Machines
  - Multi-tape, Non-deterministic – it does not matter
- Unrestricted grammars
  - Equivalent to Turing machines (recursively enumerable languages)
- Seen before: Context-free grammars
  - Equivalent to pushdown automata
- Regular grammars
  - Equivalent to finite automata and regular expressions

Undecidability

- Not all problems can be solved efficiently (P vs NP)
- Some problems cannot be solved at all! (halting problem)

Watch this great animation:
https://www.youtube.com/watch?v=92WHN-pAFCs