Languages and Machines (Module 7 TCS+IAM)

L&M 7: Beyond Context-Free Languages: Pumping Lemma for CFG and Turing Machines

Ch 7:4-5; Ch 8:1-7

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Lecture 7

Context-Free Grammars and Pushdown Automata

Parsing Algorithm (parse tree)

- Naive parsing: inefficient
- CYK parsing: dynamic programming, cubic in input length

Chomsky normal form

- Rules $A \to BC$, $A \to a$ or $S \to \lambda$
- Systematically constructed through transformation

Greibach normal form

- Rules $A \to a\,B_1\,B_2\,\cdots\,B_n$ or $S \to \lambda$
- Systematically constructed through transformation

Push-down automata

- Finite automata + unbounded stack (push/pop)
- Acceptance by “final state” and “empty stack”
- Recognize precisely the context-free languages

Contents

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Non-context-free languages

- Not all languages (sets of strings over Σ) are context-free
- How can we prove that a given language is not context-free?
- Answer: *pumping lemma*, similar to those for regular languages

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Reminder: pumping lemma regular languages

If a language X is regular, then

- there exists a $k \geq 1$, such that
- for every $z \in X$ with $|z| > k$,
- there exist $u, v, w \in \Sigma^*$, such that $z = uvw$, $|uv| \leq k$, $|v| > 0$
- and for every $i \geq 0$, we have $uv^i w \in X$

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Pumping lemma context-free languages

If a language X is context-free,

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- for every $z \in X$ with $|z| > k$, there
- exist $u, v, w, x, y \in \Sigma^*$, with $z = uvwxy$, $|vwx| \leq k$, $|vx| > 0$
- and for every $i \geq 0$, we have $uv^i wx^i y \in X$

Using the context-free pumping lemma

Contraposition of the context-free pumping lemma

A set of strings $X \subseteq \Sigma^*$ is *not* context-free if

- *for all* $k \geq 1$
- *at least one* $z \in X$ exists such that $|z| > k$ and
- *for all* $u,v,w,x,y \in \Sigma^*$ with $z{=}uvwxy$, $|vwx| \leq k$, and $|vx| > 0$
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The Pumping Lemma can be used, to prove that languages are not context-free, e.g.:

- $\{a^i b^i c^i \mid i \geq 0\}$
- $\{a^i b^j a^i b^j \mid i,j \geq 0\}$

Example of the context-free pumping game

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     - Let $\{\alpha, \beta, \gamma\} \subseteq \{a, b, c\}$ such that $v = \alpha^m$ and $x = \beta^n$ and $\gamma \notin \{\alpha, \beta\}$.
     - Then uv^2wx^2y contains the same number of γ's as z does, but more α's and/or $\beta's$
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So by the pumping lemma, X is not context-free.

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Closure properties context-free languages

Reminder: closure properties regular languages

If X and Y are regular languages over alphabet Σ, then also

- Union: $X \cup Y$
- Concatenation: $XY = \{vw \mid v \in X, w \in Y\}$
- Kleene star: $X^* = \{v_1 \cdots v_n \mid \forall i \in [1, n] : v_i \in X\}$
- Intersection: $X \cap Y$
- Complement: $\Sigma^* - X$

In other words, regular languages are *closed under* these operations.

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Theorem: Non-closure properties context-free languages

CF languages are *not* closed under intersection & complement

Proof of (non-)closure properties

Given grammars $G_i = \langle V_i, \Sigma_i, P_i, S_i \rangle$ with $L(G_i) = X_i$ $(i = 1, 2)$

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- Not closed under intersection
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  - Closed under complement implies closed under intersection
  - Contradiction, so CF languages are not closed under complement

Another closure property

Theorem

Intersection of a regular and a context-free language is context-free

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Proof (another application of parallel composition!)

Construct *product* of a PDA M and a DFA N:

$$M \times N = \langle Q_M \times Q_N, \Sigma_M \cup \Sigma_N, \Gamma_M, \delta, (q_{M,0}, q_{N,0}), F_M \times F_N \rangle$$

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- $q_1 \xrightarrow{\lambda \;\; B/C} q_1'$ generates $(q_1, q_2) \xrightarrow{\lambda \;\; B/C} (q_1', q_2)$

Now $L(M \times N) = L(M) \cap L(N)$

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- Application: $L = \{ww \mid w \in \{a, b\}^*\}$ is not context-free
- Proof (by contradiction): Suppose that L is context-free.
  - Then $L \cap a^*b^*a^*b^* = \{a^i b^i a^i b^i \mid i \geq 0\}$ is as well (theorem).
  - But this is not the case (pumping lemma): Contradiction
  - So L is not context-free either.

Side note: Deterministic Context-Free Languages

L is deterministic context-free if it is accepted by a deterministic PDA.
The situation for the class of DPDAs is very different:

- DPDAs are weaker than PDAs
- DPDAs are NOT closed under union
- DPDAs are closed under complement
  (this fact is very difficult to prove)

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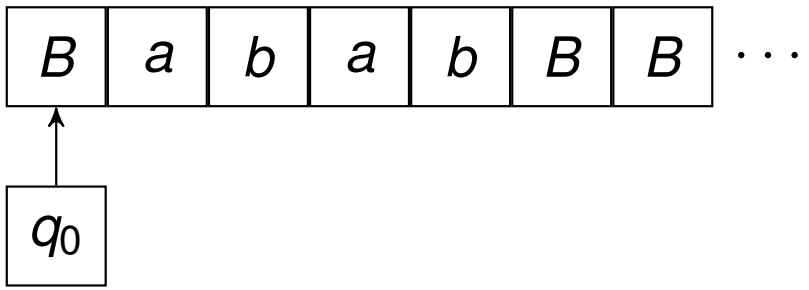
  (this fact is very difficult to prove)
- 2002: Geraud Senizergues receives Gödel Award for algorithm that decides if $\mathcal{L}(M_1) = \mathcal{L}(M_2)$ for deterministic PDAs M_1, M_2.
- Provable fact: No algorithm exists that
  decides whether $\mathcal{L}(M_1) = \mathcal{L}(M_2)$ for arbitrary PDAs M_1, M_2.

Contents

Turing Machines

- A Turing machine has an *infinite tape* for memory
- Configuration:



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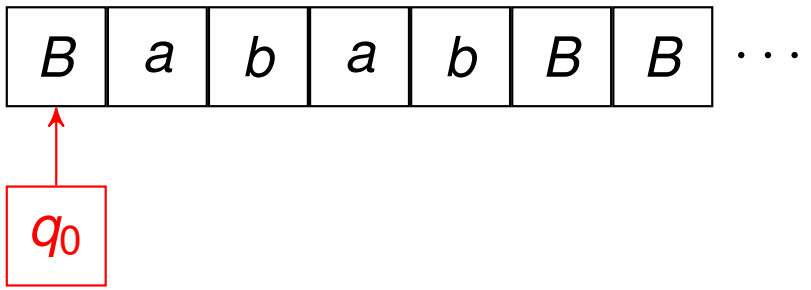

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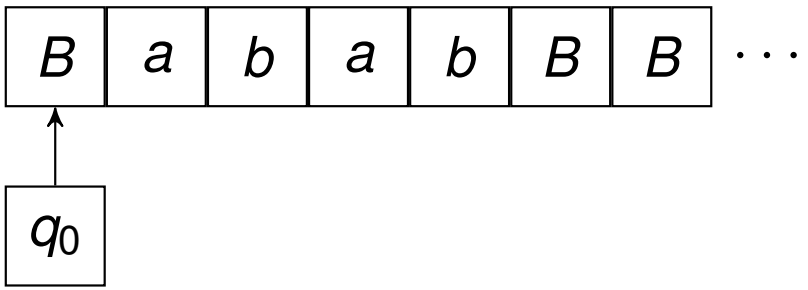

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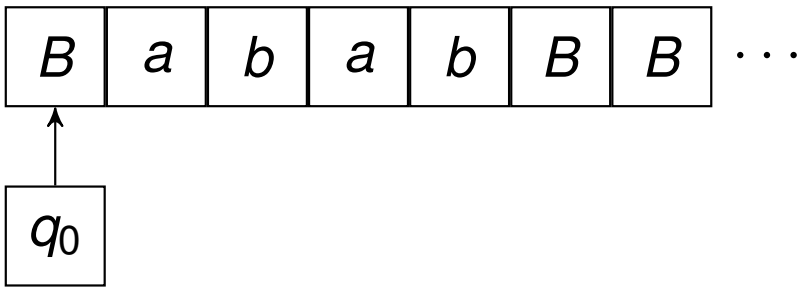

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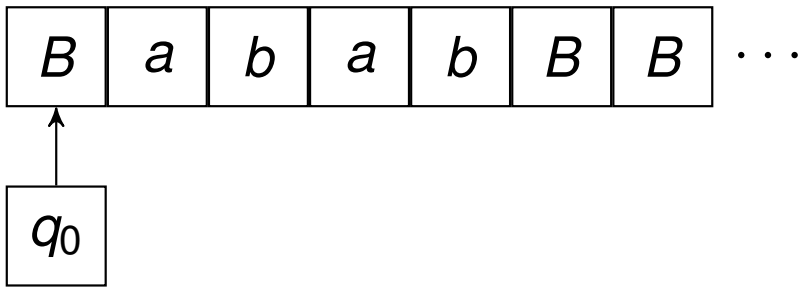

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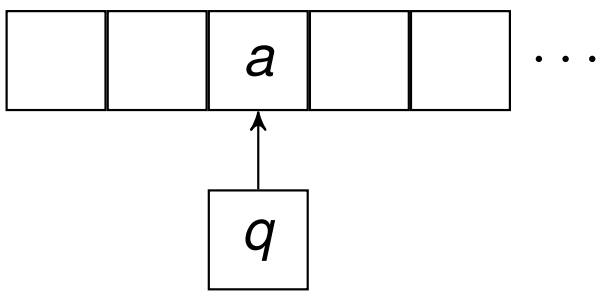

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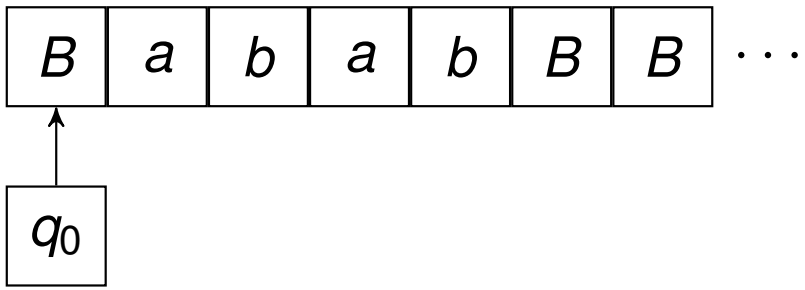

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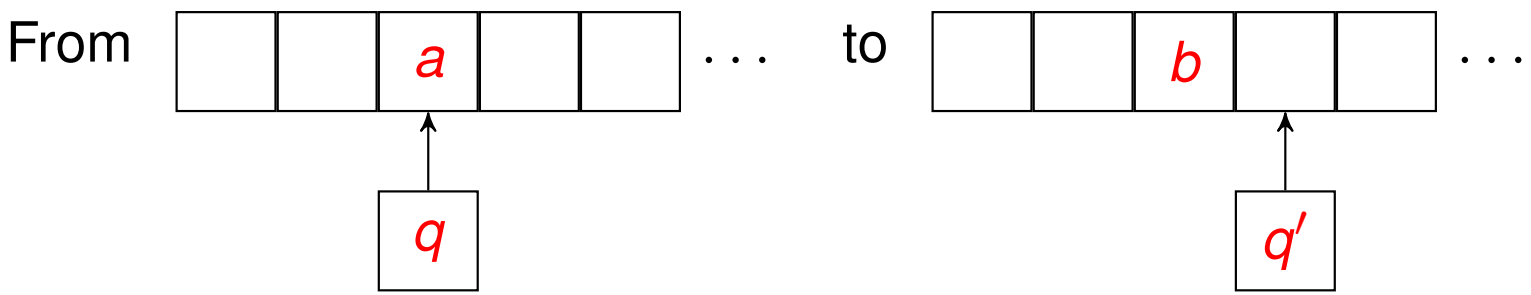

Functioning of a Turing machine

Definition: Turing Machine

A *Turing machine* is a tuple $\langle Q, \Sigma, \Gamma, \delta, q_0, F \rangle$ with

- Q a set of states
- Γ the tape alphabet, with special *blank* symbol $B \in \Gamma$
- Σ the input alphabet, with $\Sigma \subseteq \Gamma \setminus \{B\}$
- $\delta : Q \times \Gamma \rightharpoonup \Gamma \times \{L, R\} \times Q$ the transition relation (deterministic)
- $q_0 \in S$ start state, $F \subseteq Q$ set of final states.

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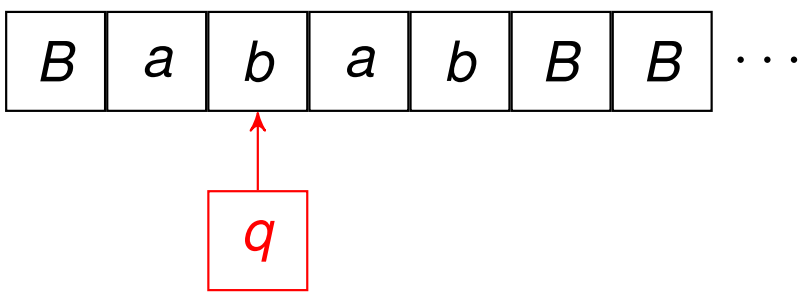
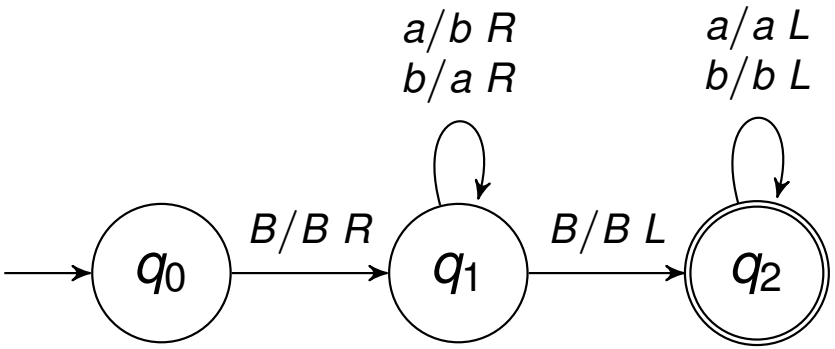


Turing machine: Example 1

$\Sigma = \{a, b\}, \Gamma = \{a, b, B\}$



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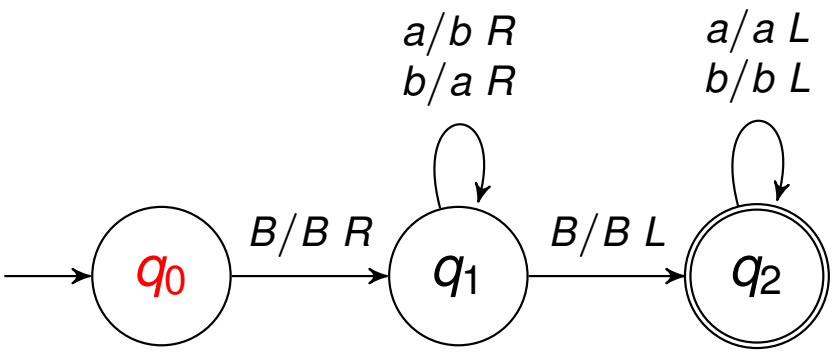

$q_0 BababB$

$\vdash$

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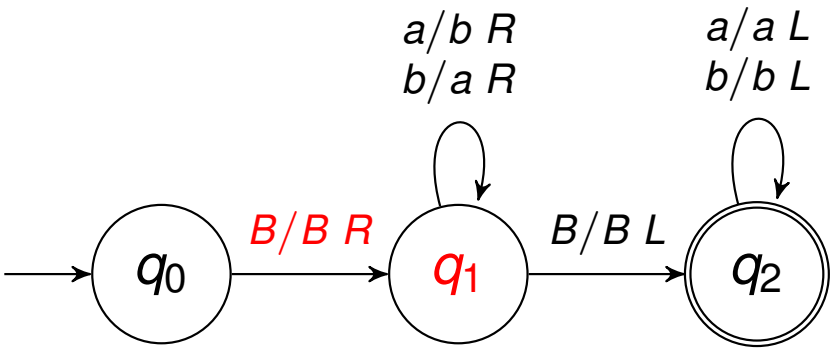

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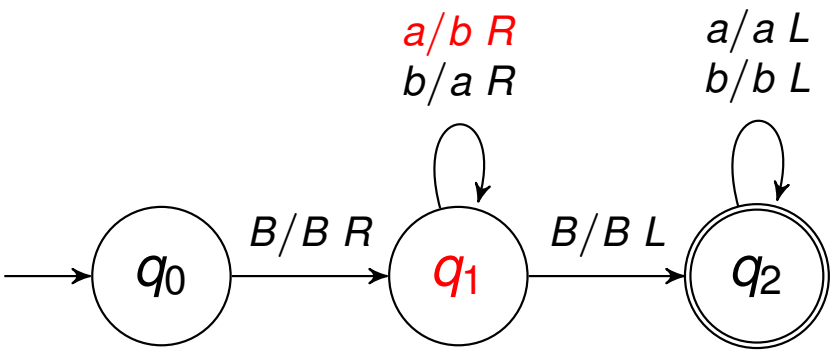

$q_0 BababB$
$\vdash B q_1 ababB$
$\vdash B b q_1 babB$
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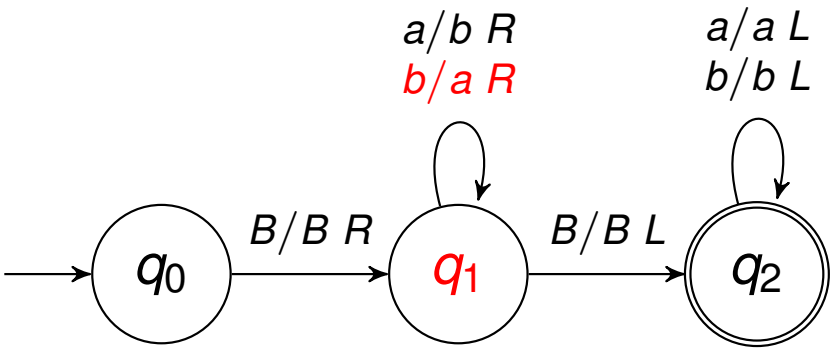

$q_0 BababB$
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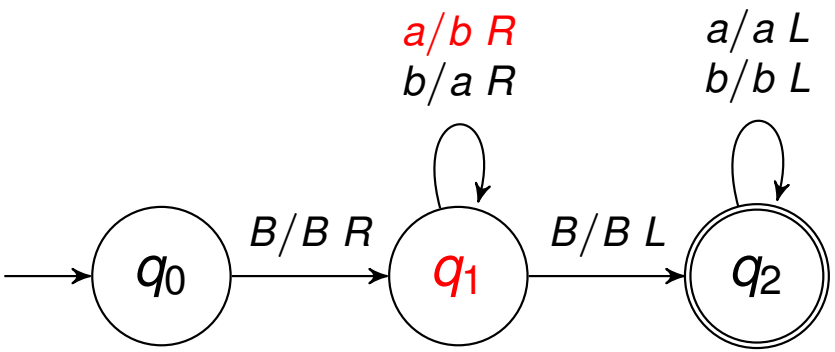

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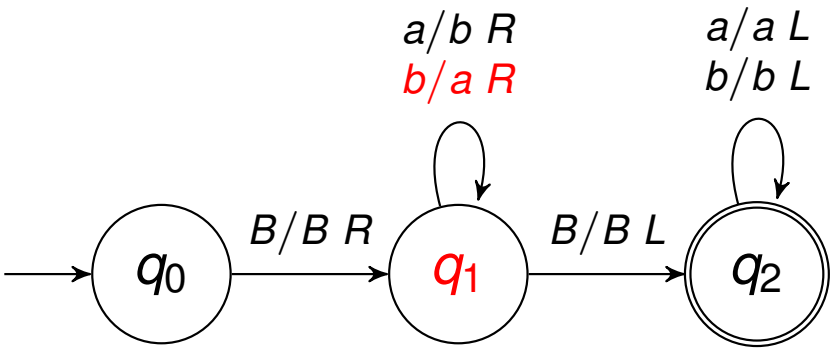

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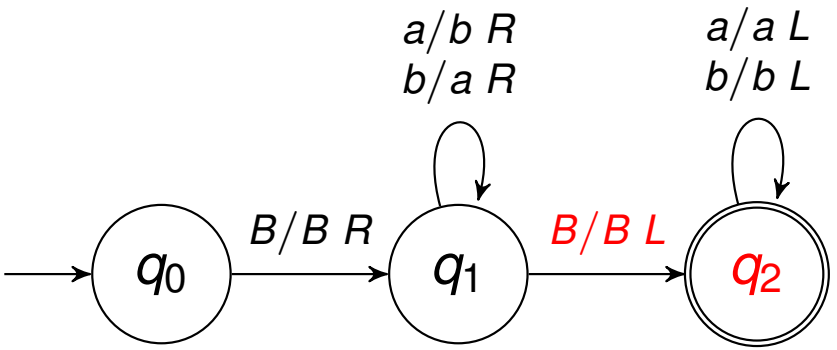

$q_0 BababB$
$\vdash Bq_1 ababB$
$\vdash Bbq_1 babB$
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$\vdash Bbabq_1 bB$
$\vdash Bbabaq_1 B$
$\vdash Bbabq_2 aB$
$\vdash$

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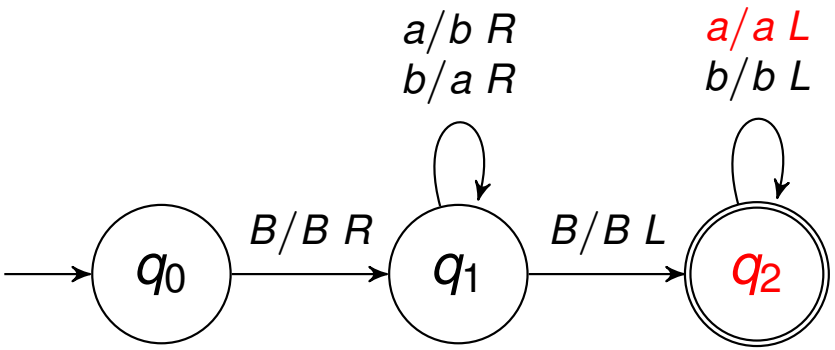

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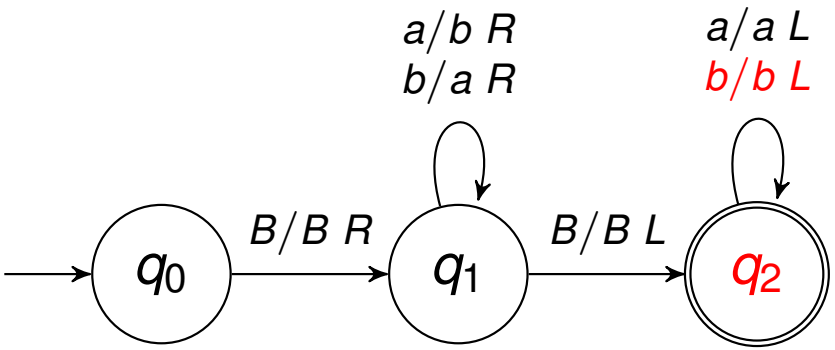

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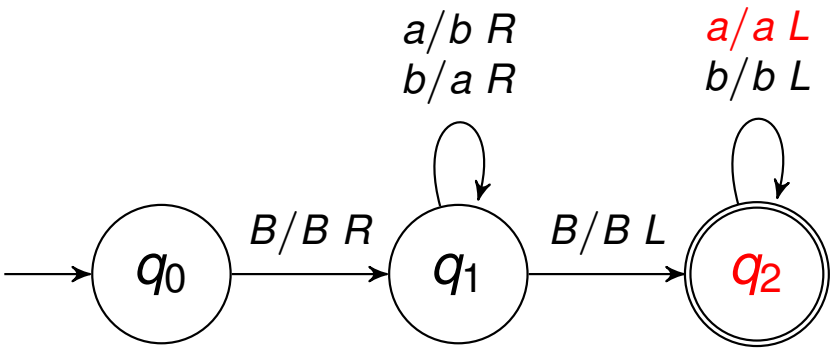

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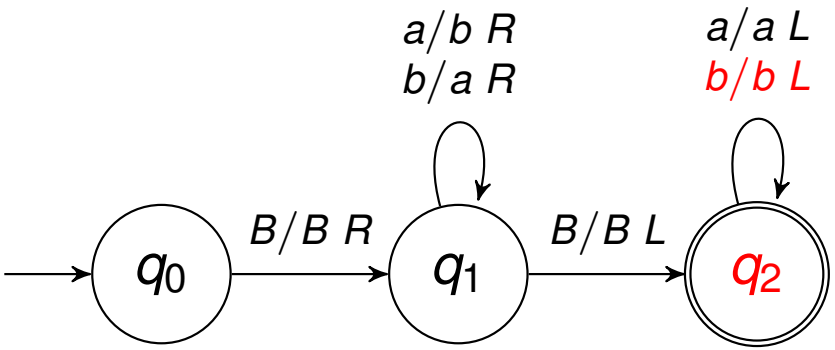

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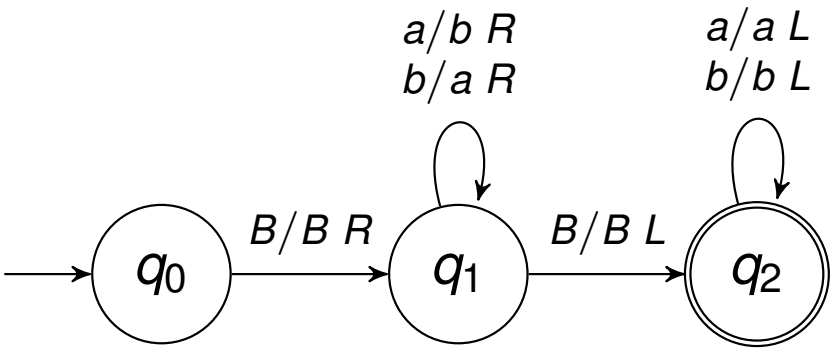

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What does this machine achieve?

Turing machine: Example 1

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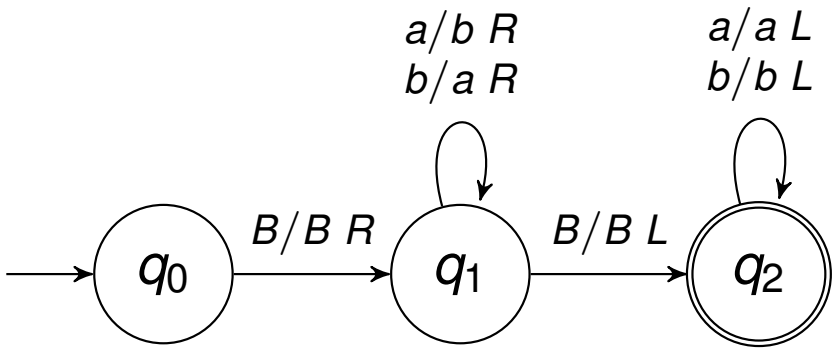

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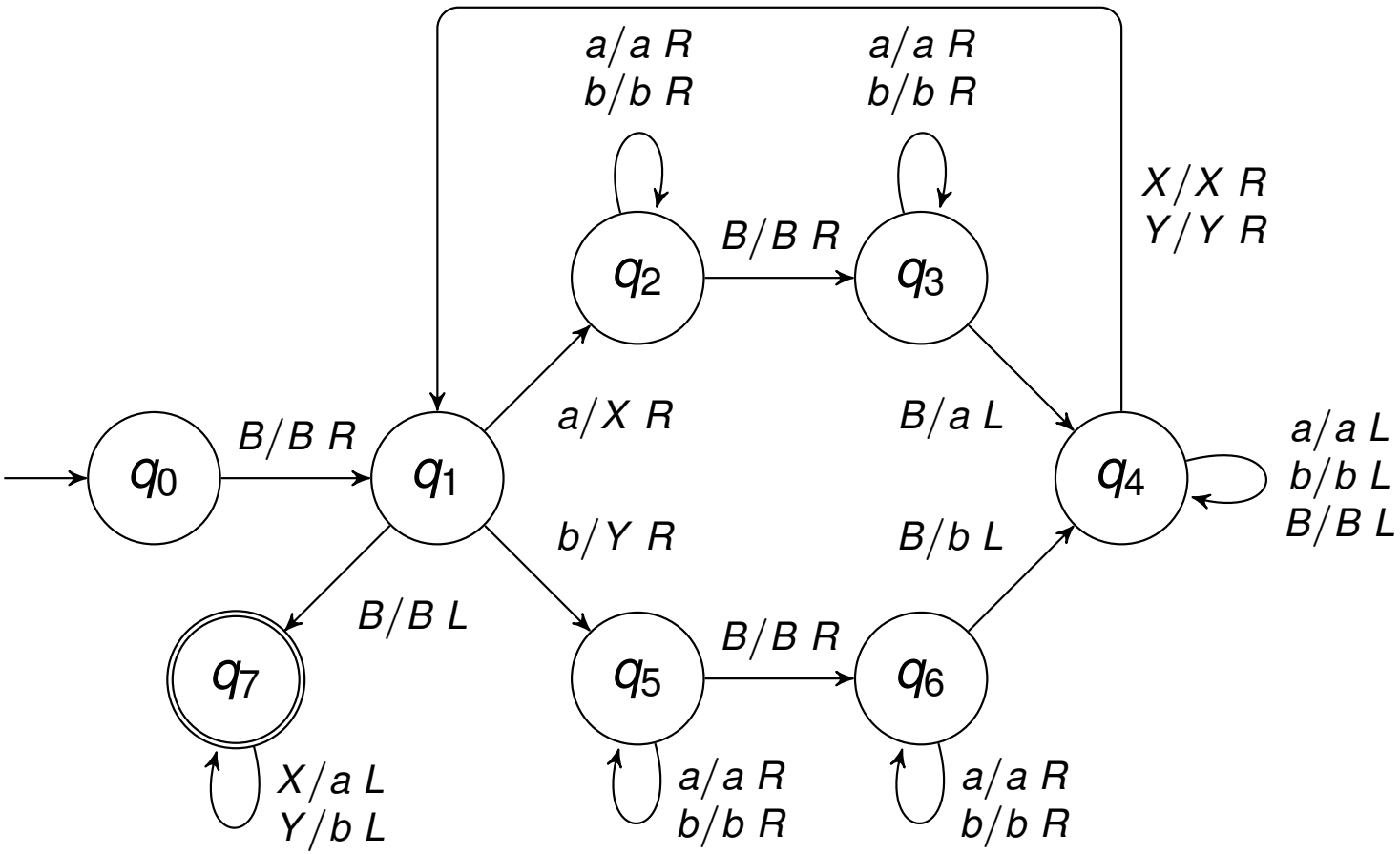
What does this machine achieve?

1. Starting from the first “real” position (not counting start)
2. It switches *a*’s and *b*’s until the first occurrence of *B*
3. The tape head returns to the start

Turing machine: Example 2

What does the following Turing machine achieve?



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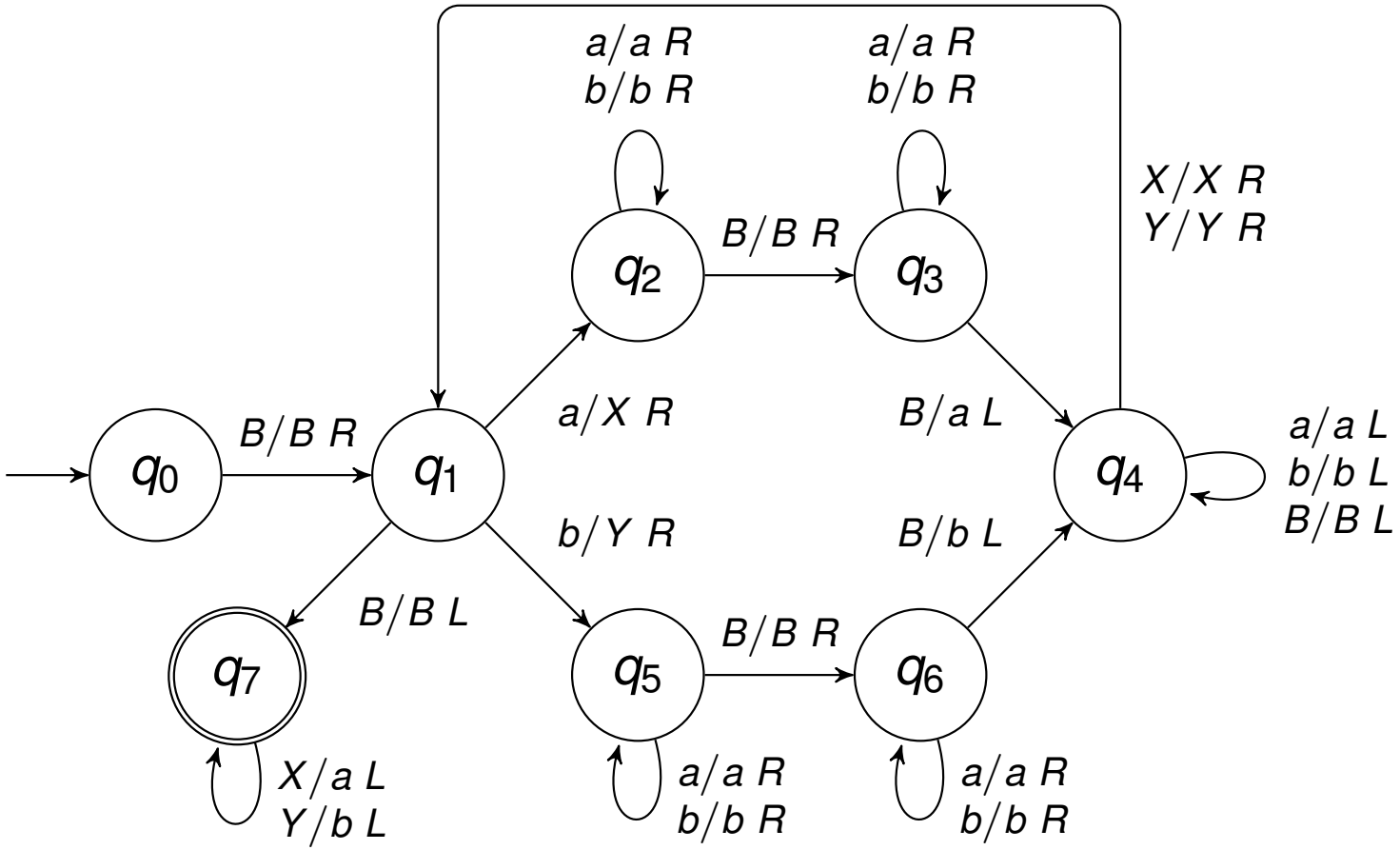

- *Copies* words over $\{a, b\}^*$
- $q_0 BaabaB \vdash \cdots Bq_1 XXYXBaabaB \vdash \cdots Bq_7 aabaBaabaB$

Acceptance by Turing machines

Further terminology:

- The word that is initially on the tape is called the *input* (from the first "real" position until the infinite series of *B*'s)
- The TM *halts* when no further transitions are possible (different from automata: just reaching a final state is not enough!)

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Definition

A Turingmachine *M accepts* a word *w* if *M* halts in a final state when starting with input *w*.

We do *not* pay attention to the contents of the tape after computation!
We do *not* care if *w* is actually read at all.

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There are three possible outcomes (for a given input):

1. The machine halts in a final state
2. The machine halts in a state that is not a final state
3. The machine never halts

Languages of Turing machines

Further terminology:

- *M recognizes* a language L if $\forall w$: M accepts $w \Leftrightarrow w \in L$
- *M decides* a language L if *moreover all computations halt*

Notation: $L(M)$ is the language that is recognized by M

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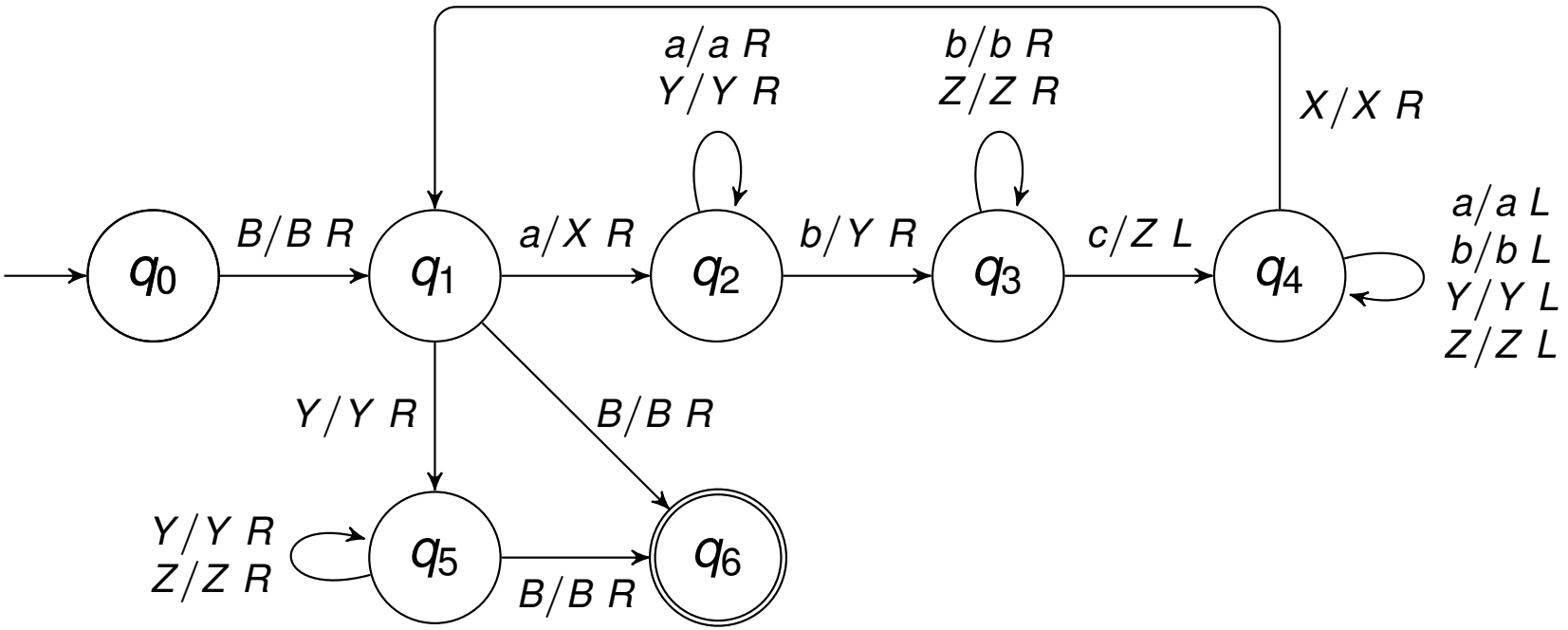
Definition: let $L \subseteq \Sigma^*$ be an arbitrary language

- L is *recursively enumerable* if L is recognized by a Turing machine
- L is *recursive* if L is decided by a Turing machine

Recursive language: Example

Which language is *decided* by the following Turing machine:



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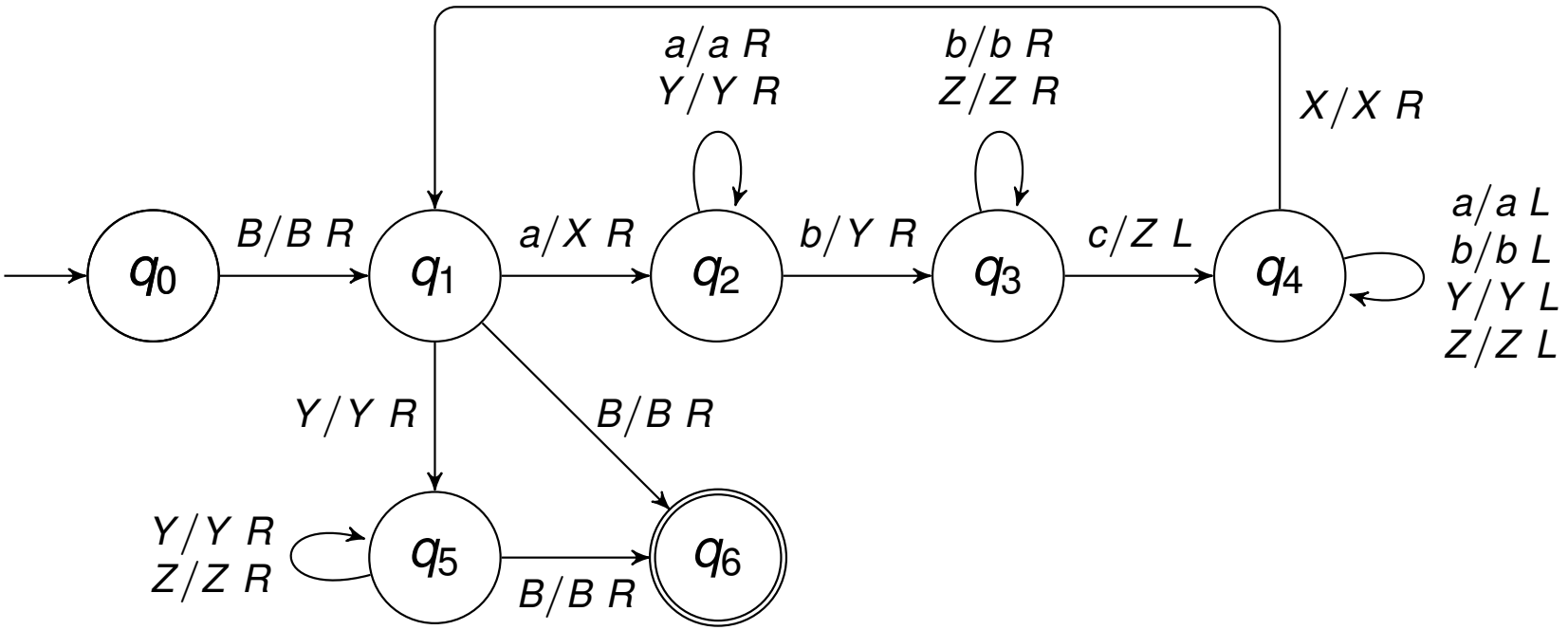

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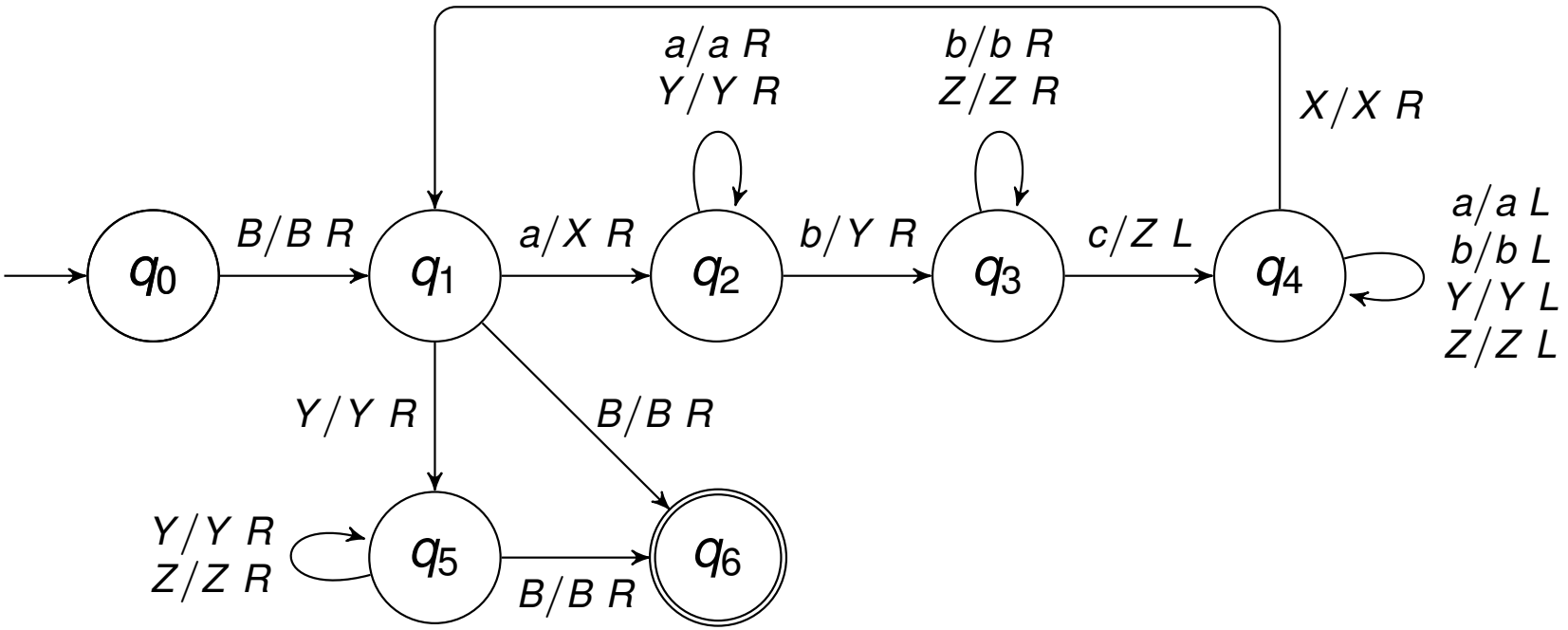

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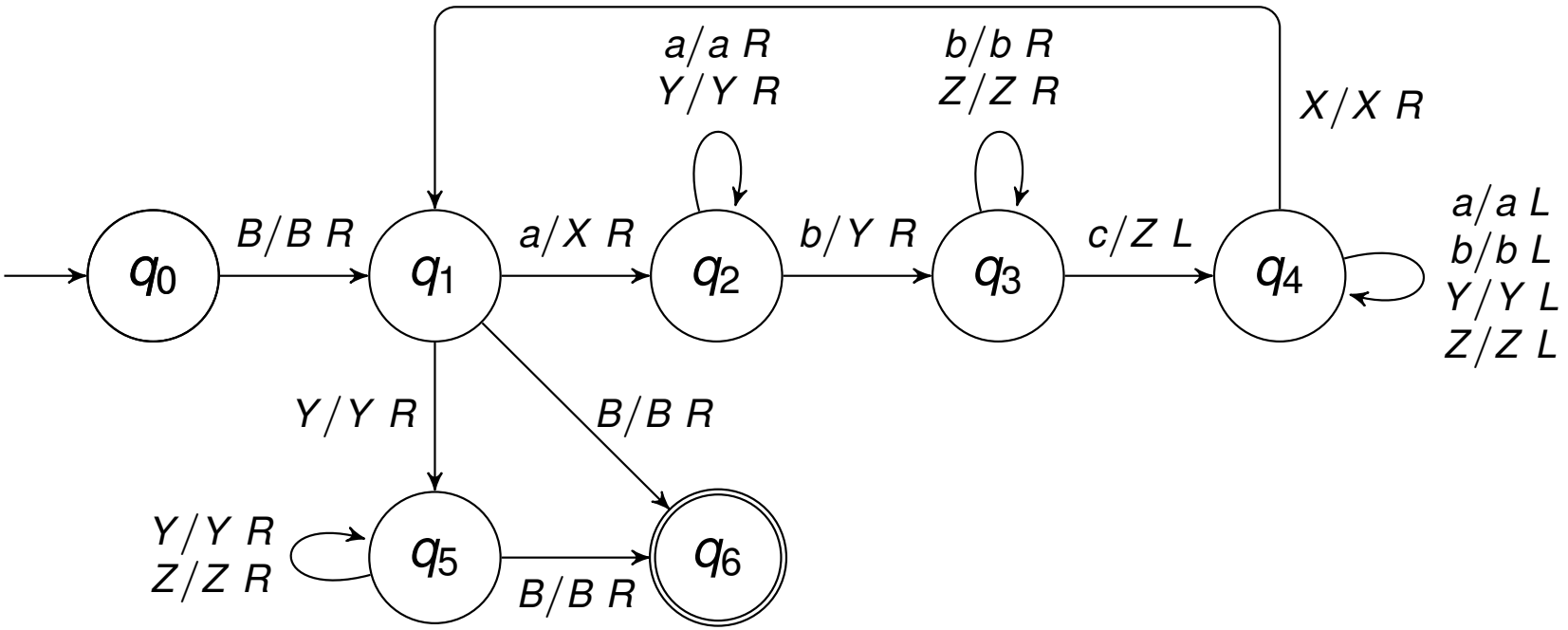

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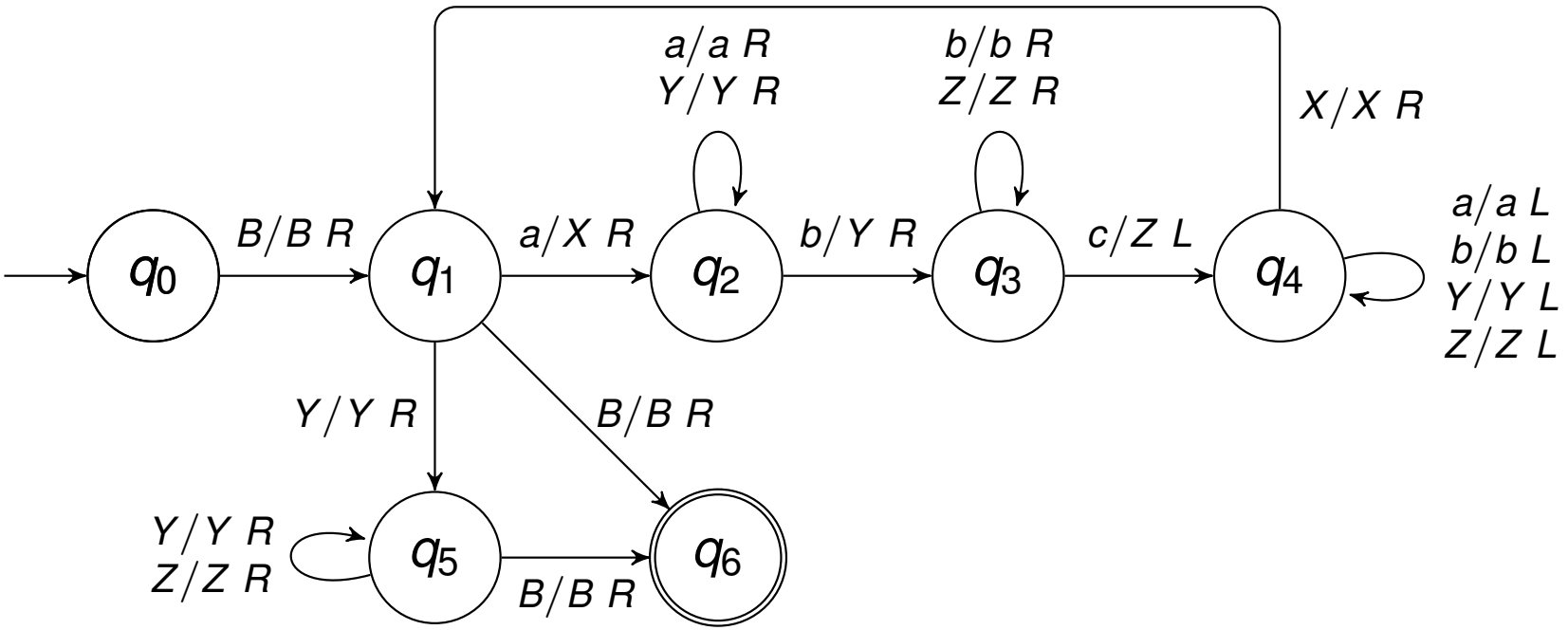

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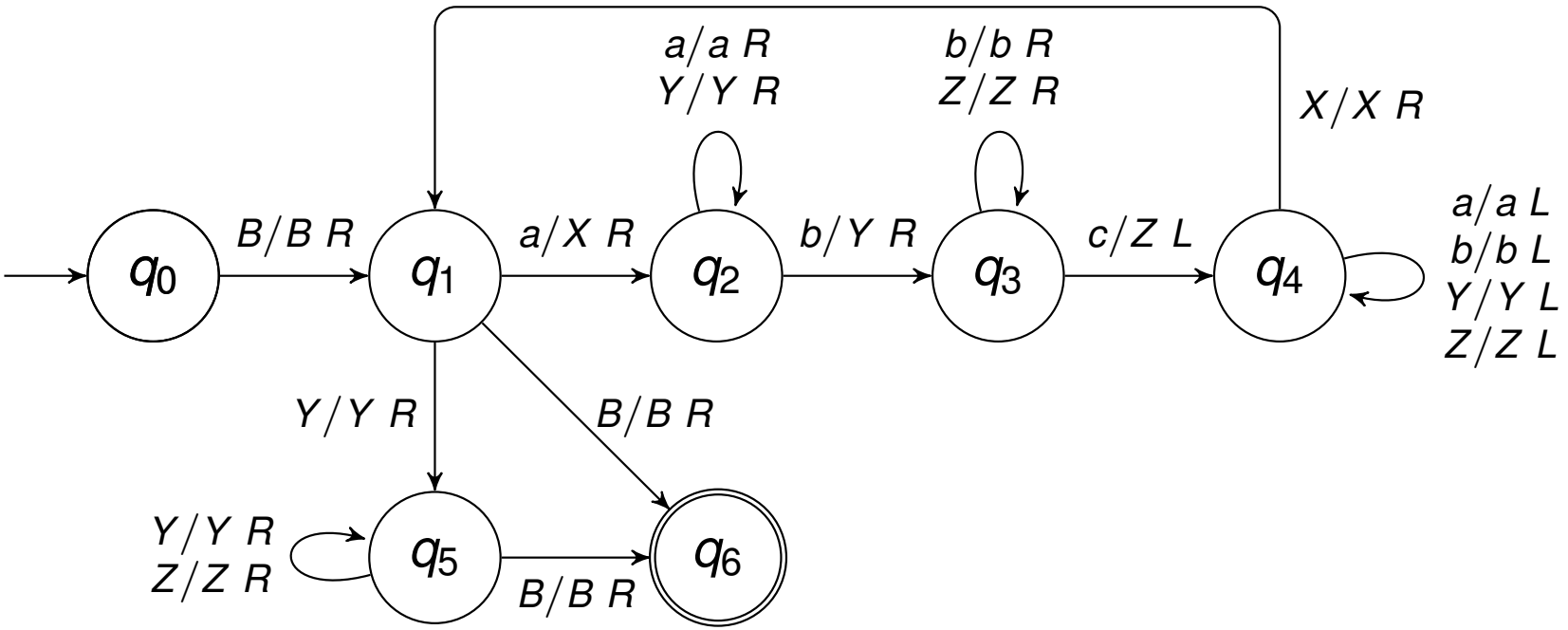

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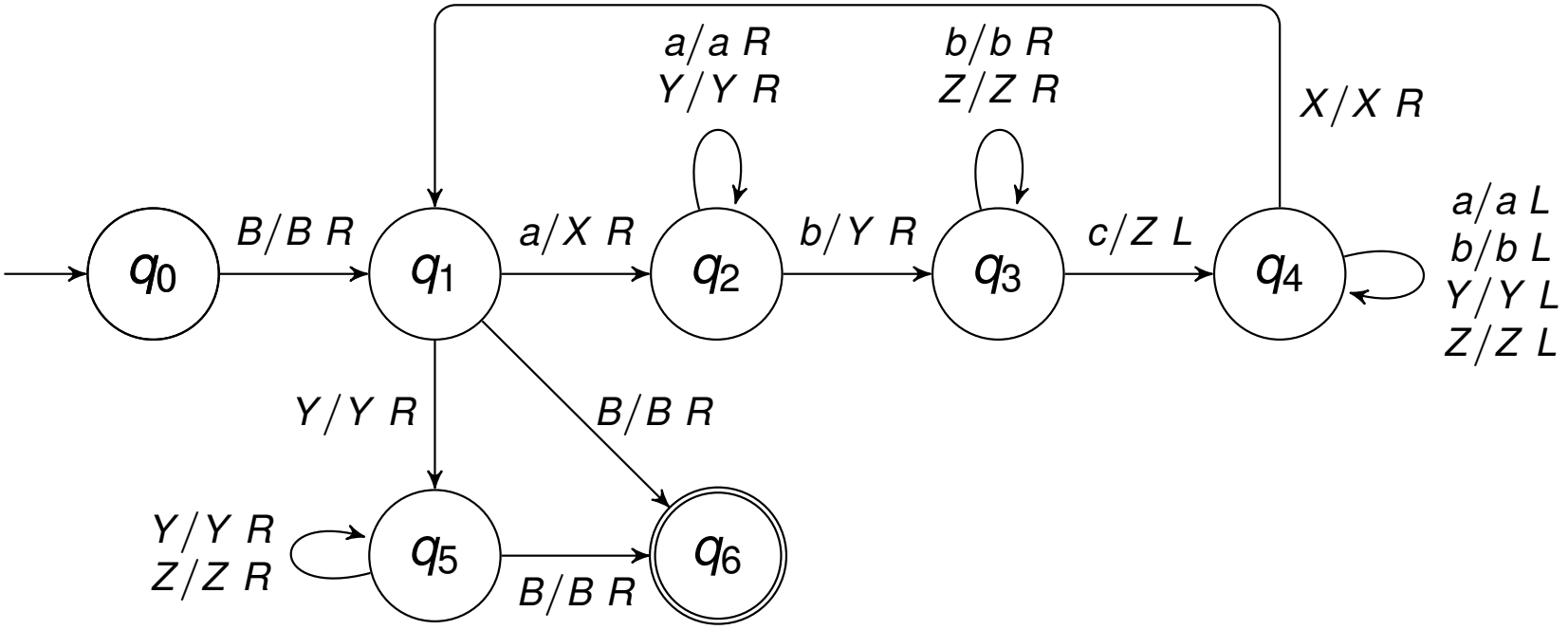

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Contents

Variatons on Turing Machines

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- Single tape
- Deterministic
- Acceptance by “halting” in a “final state”

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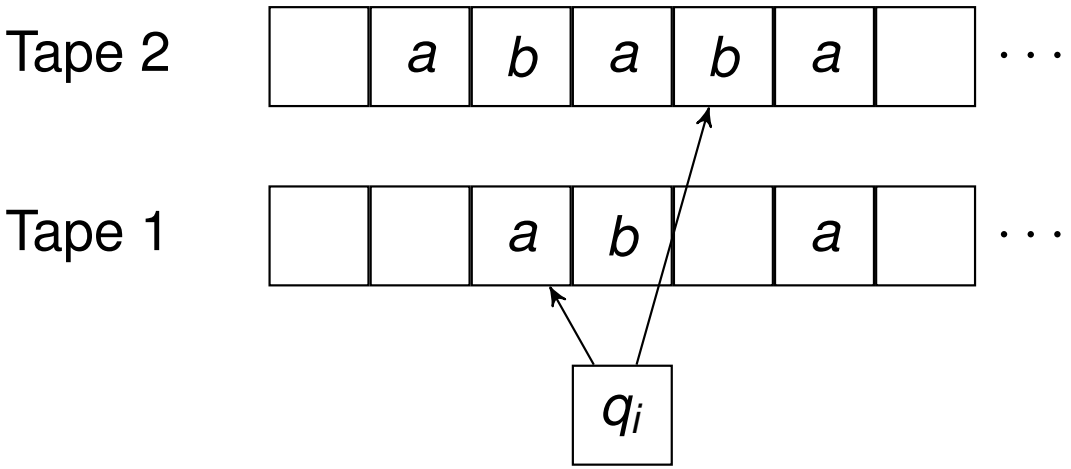
- Single tape
- Deterministic
- Acceptance by “halting” in a “final state”

The following variations are all equivalent!

- Acceptance by “halting”
- Multi-track tape
- Two-way tape
- Multiple tapes
- Non-deterministic

Playing with the tape: Multiple tapes

Instead of one tape we can also have multiple:



- Difference with multiple tracks: tape head for each tape
- Transitions $q \xrightarrow{[x_1/y_1\ d_1, x_2/y_2\ d_2]} q'$
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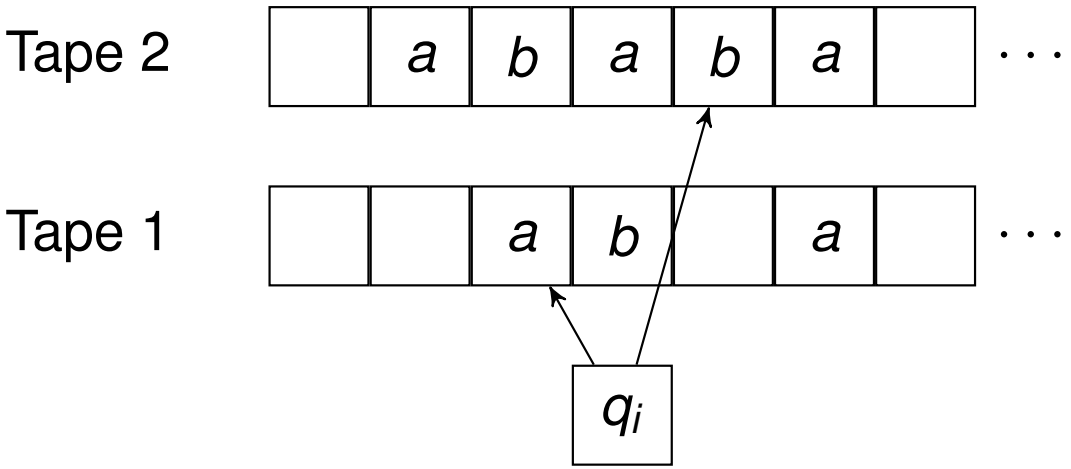

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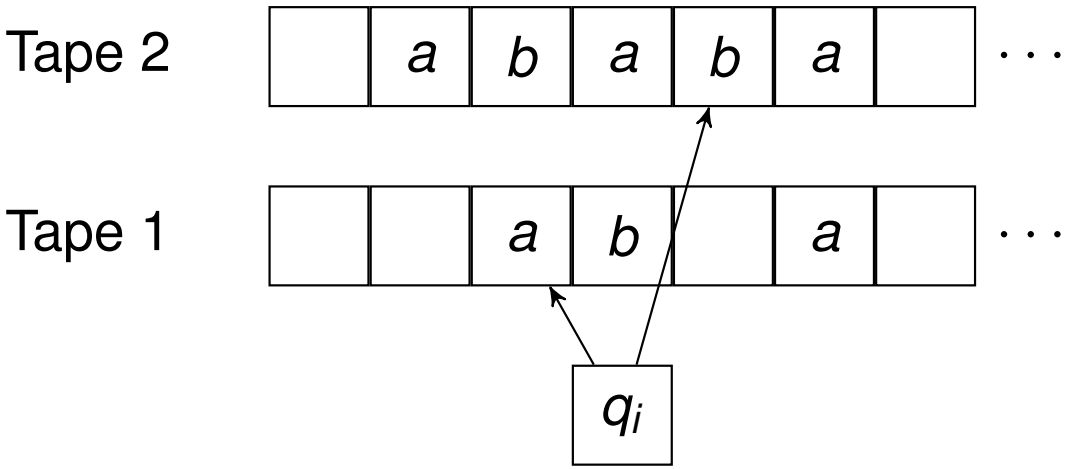

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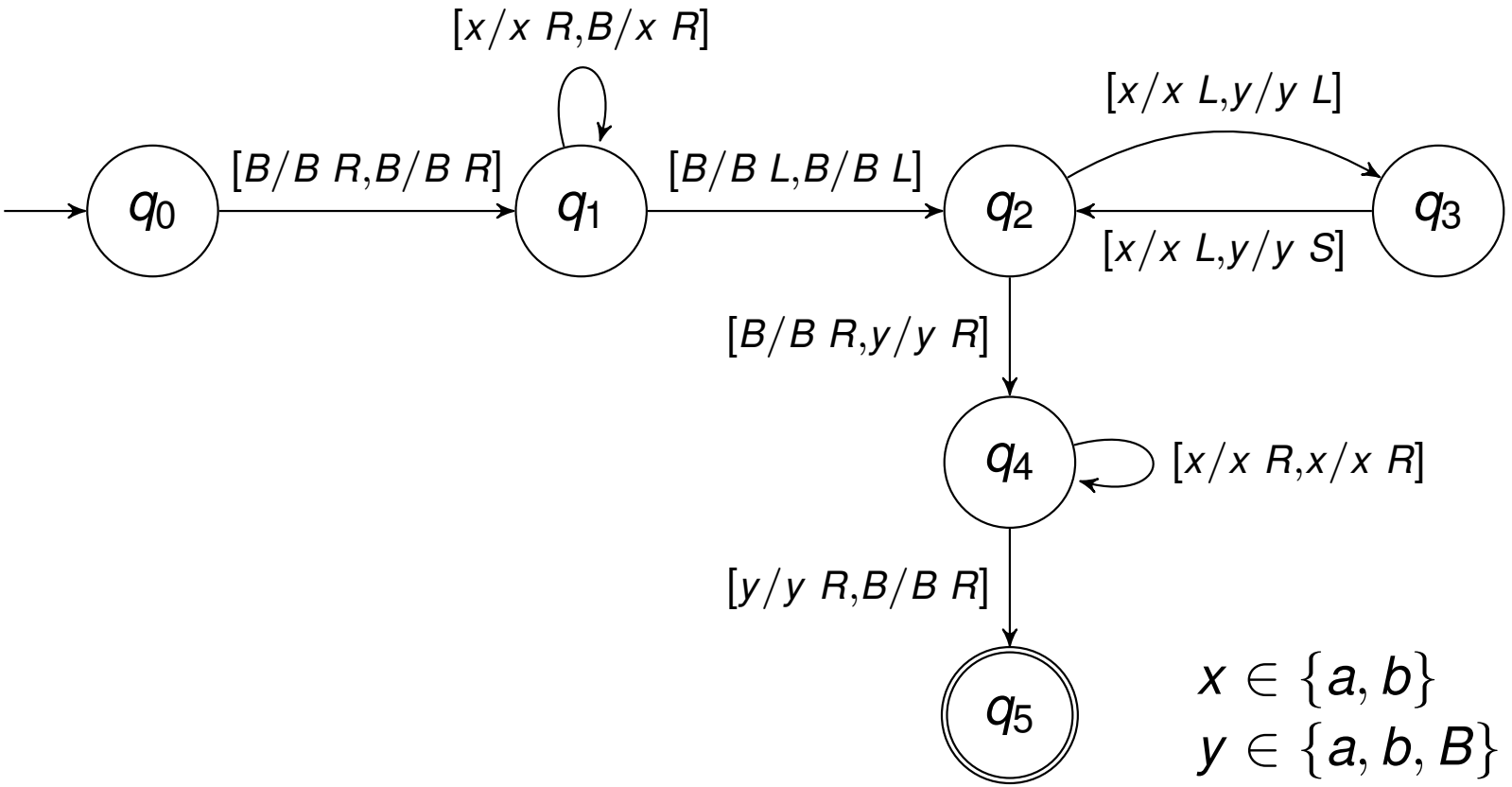
Theorem

A language is accepted by a machine with multiple tapes if and only if it is recursively enumerable.

Example of a machine with multiple tapes



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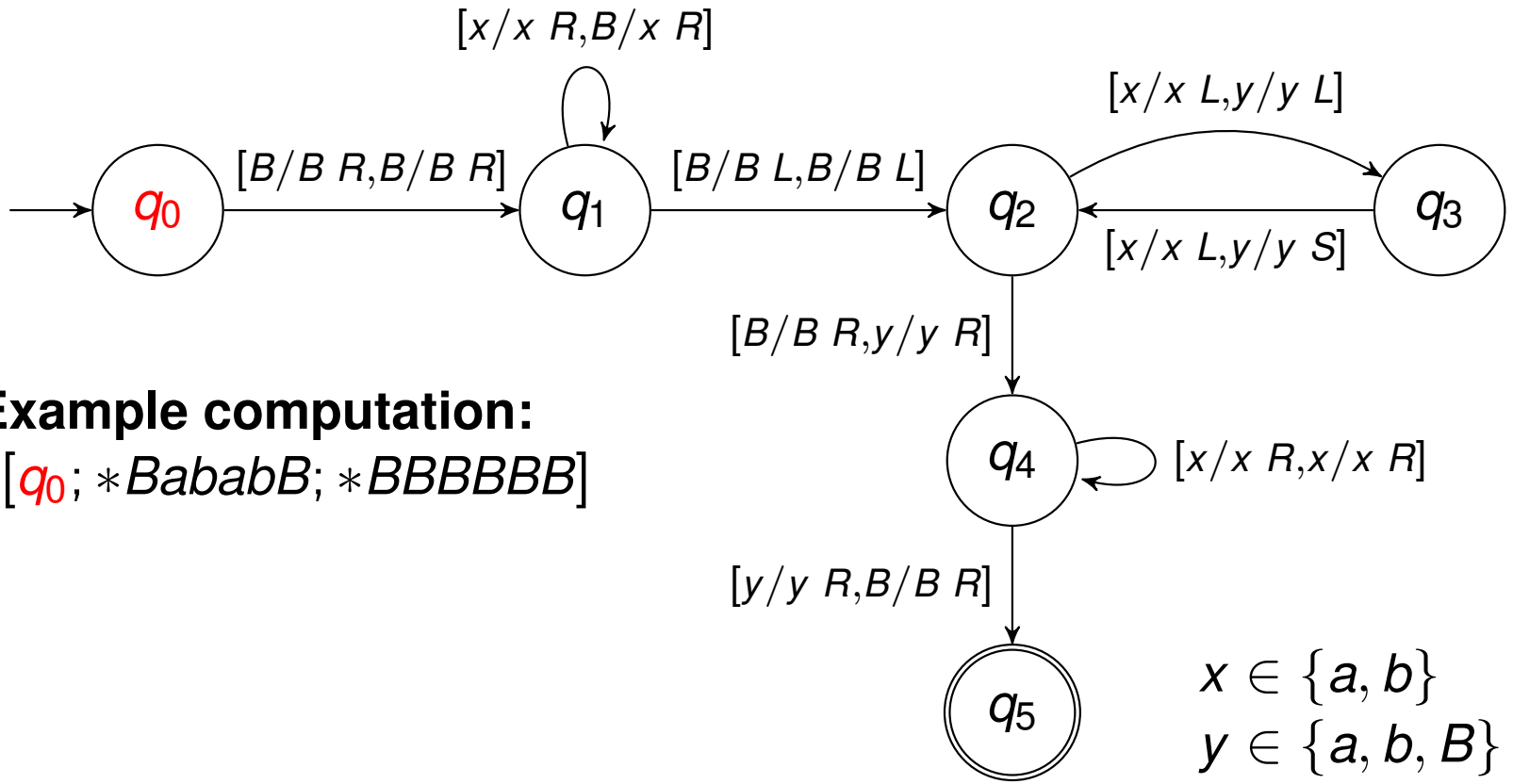

Example computation:

$[q_0; *BababB; *BBBBBB]$

$x \in \{a, b\}$

$y \in \{a, b, B\}$

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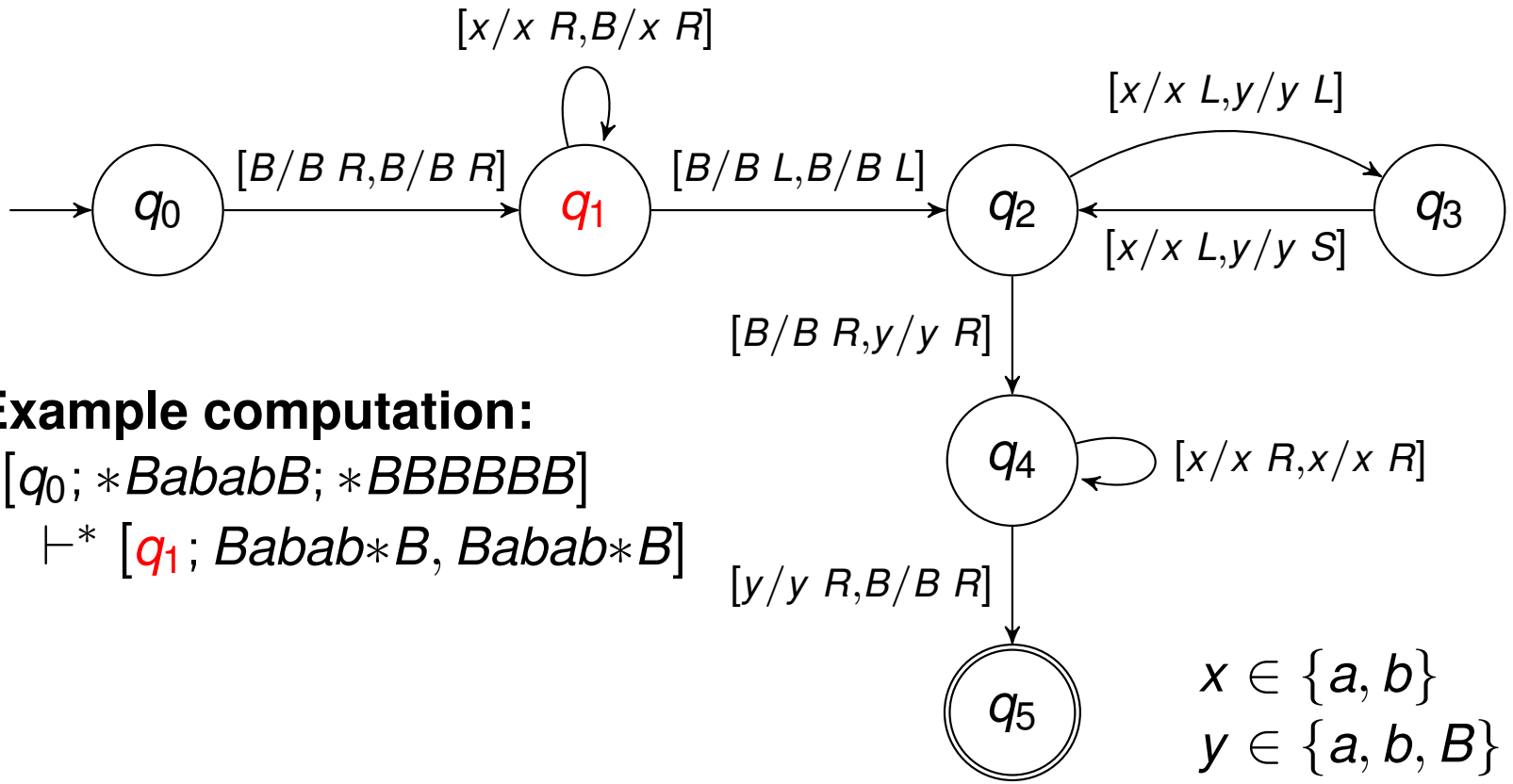

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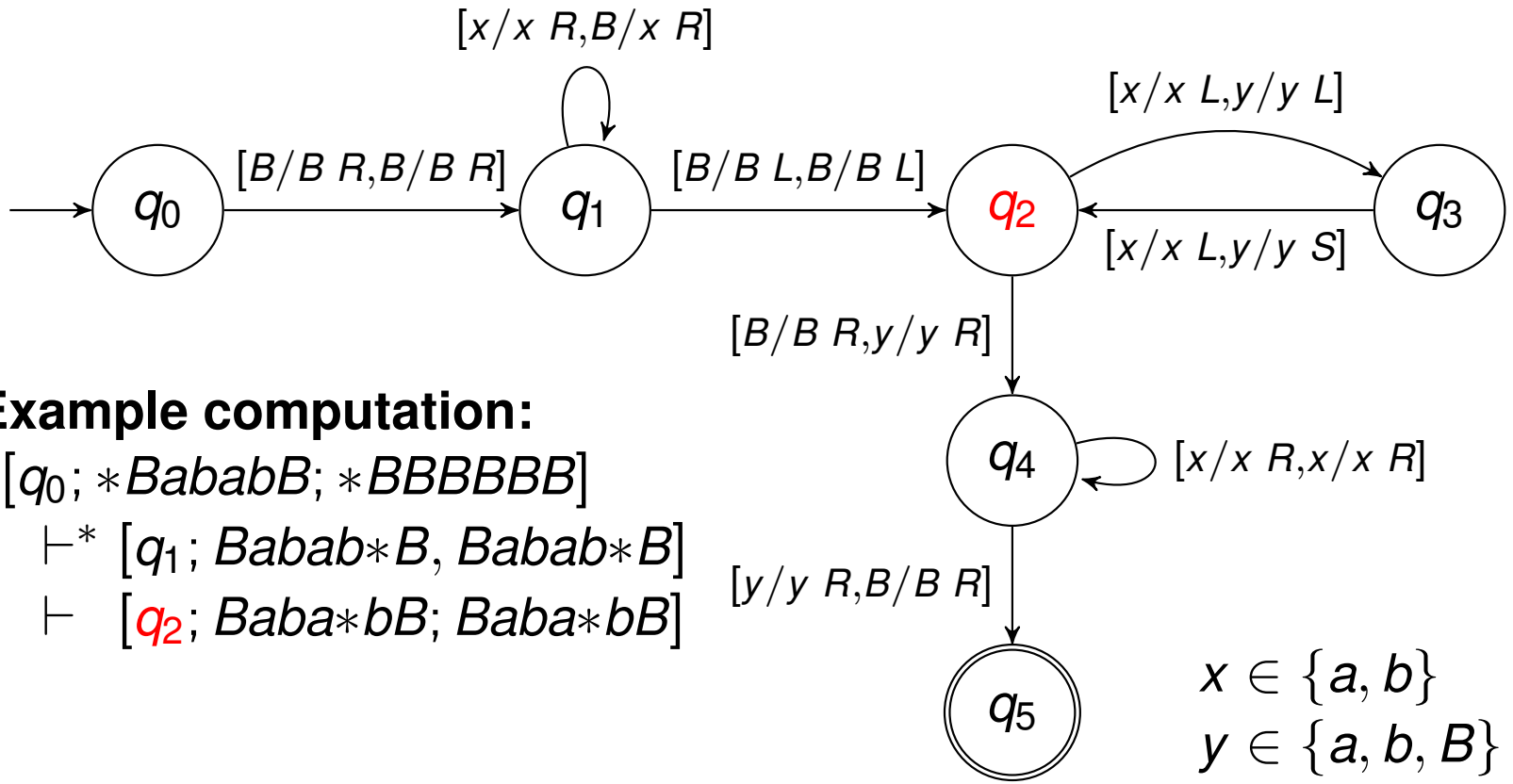

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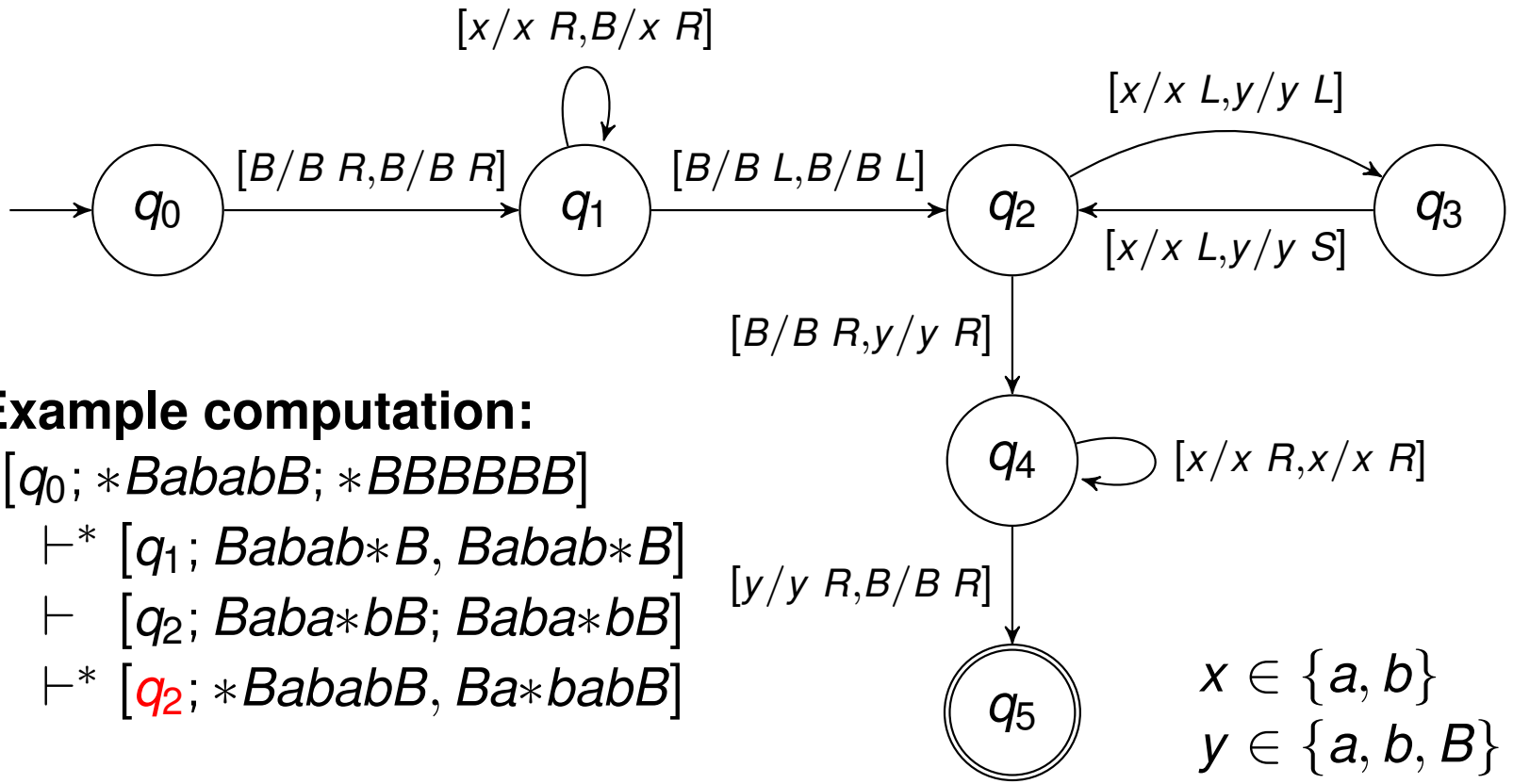

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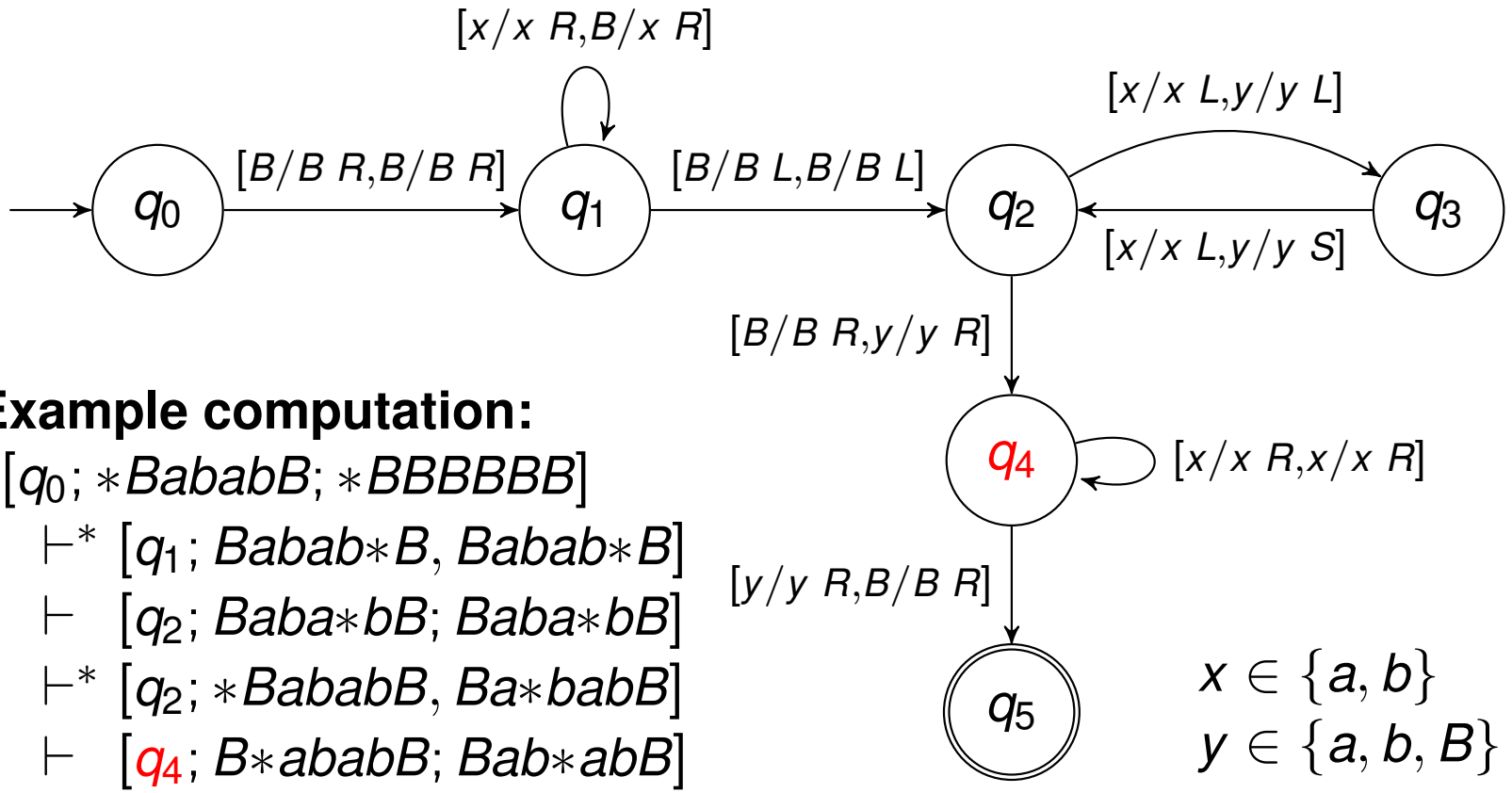

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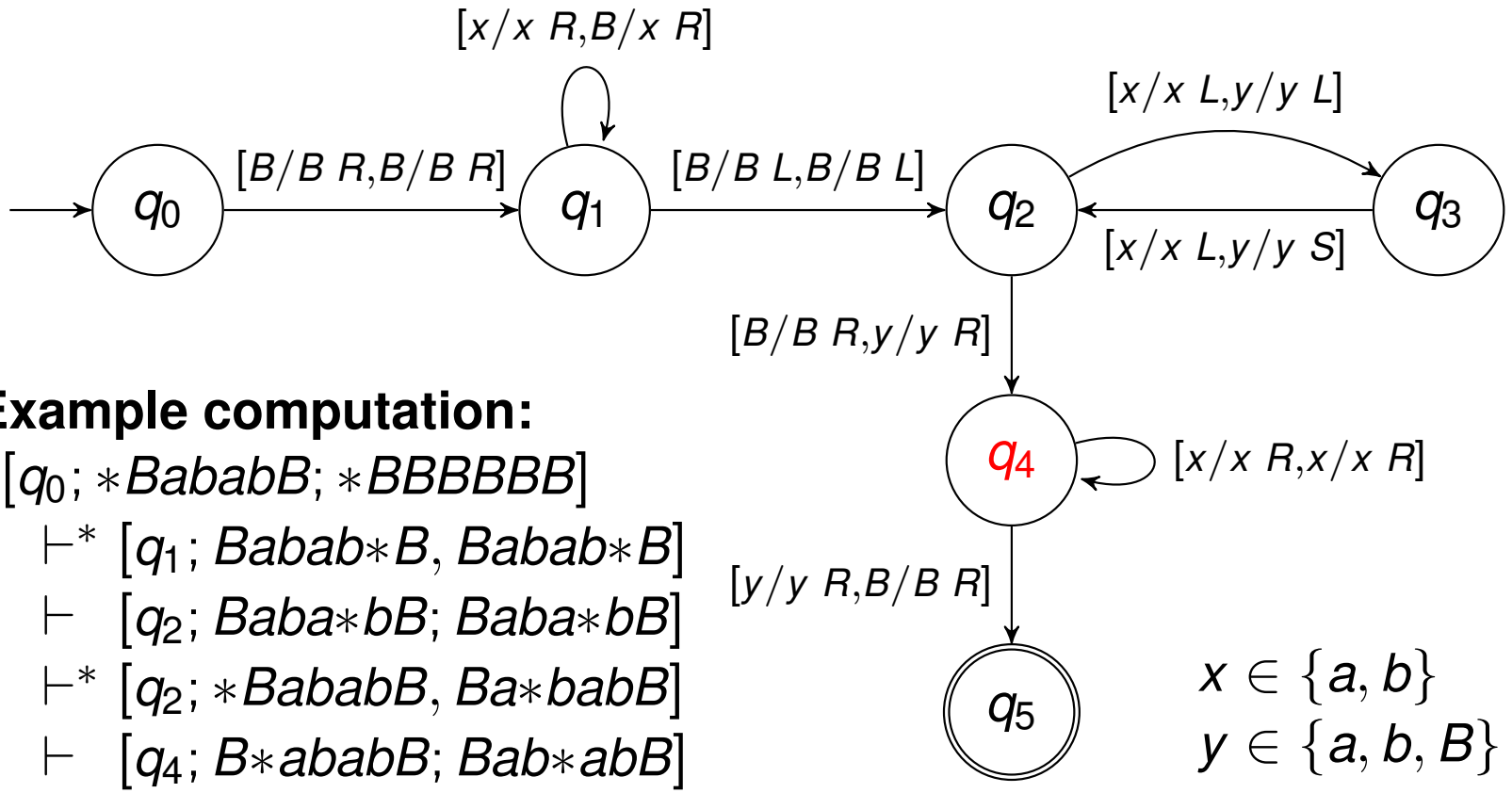

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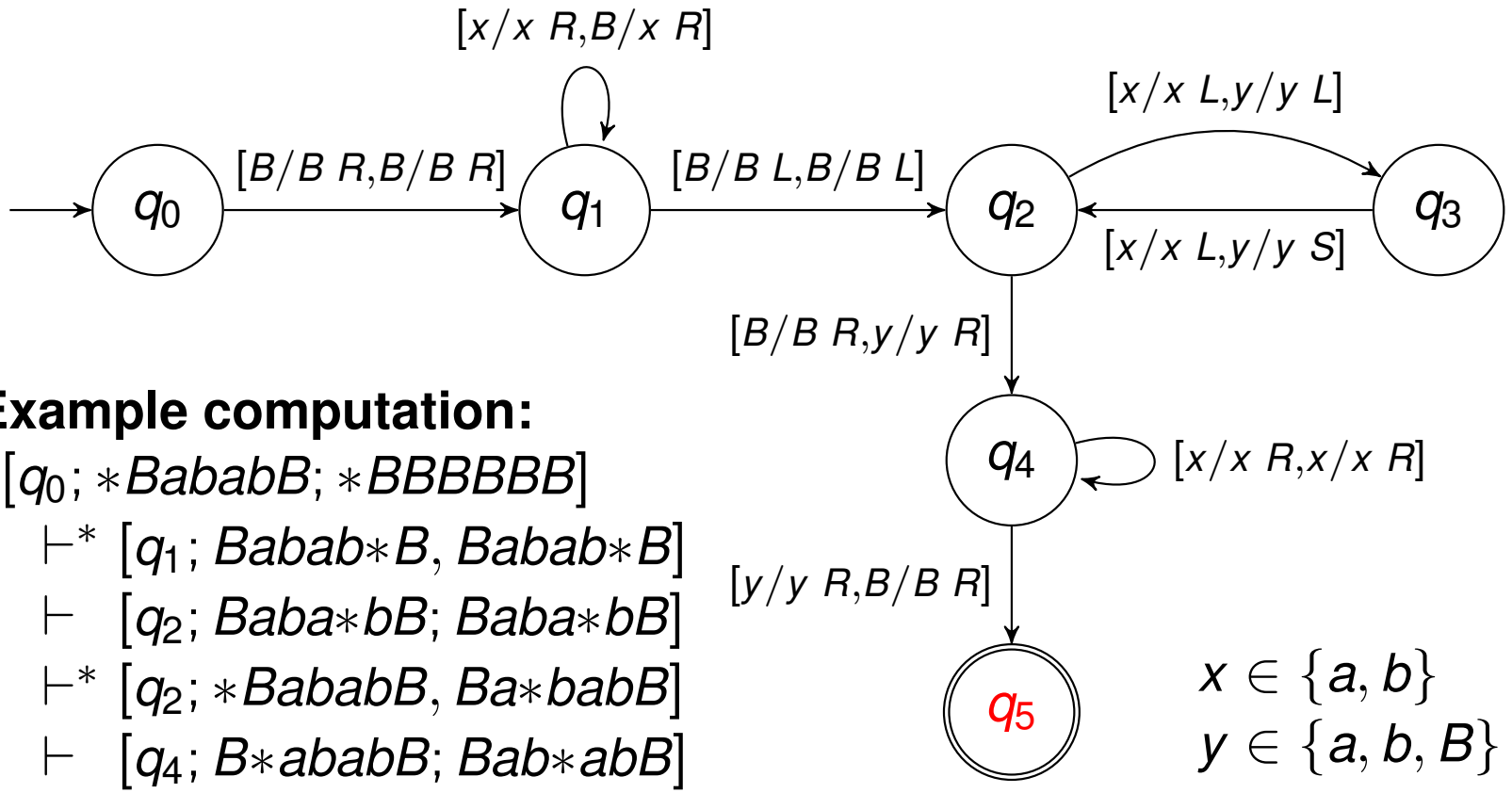

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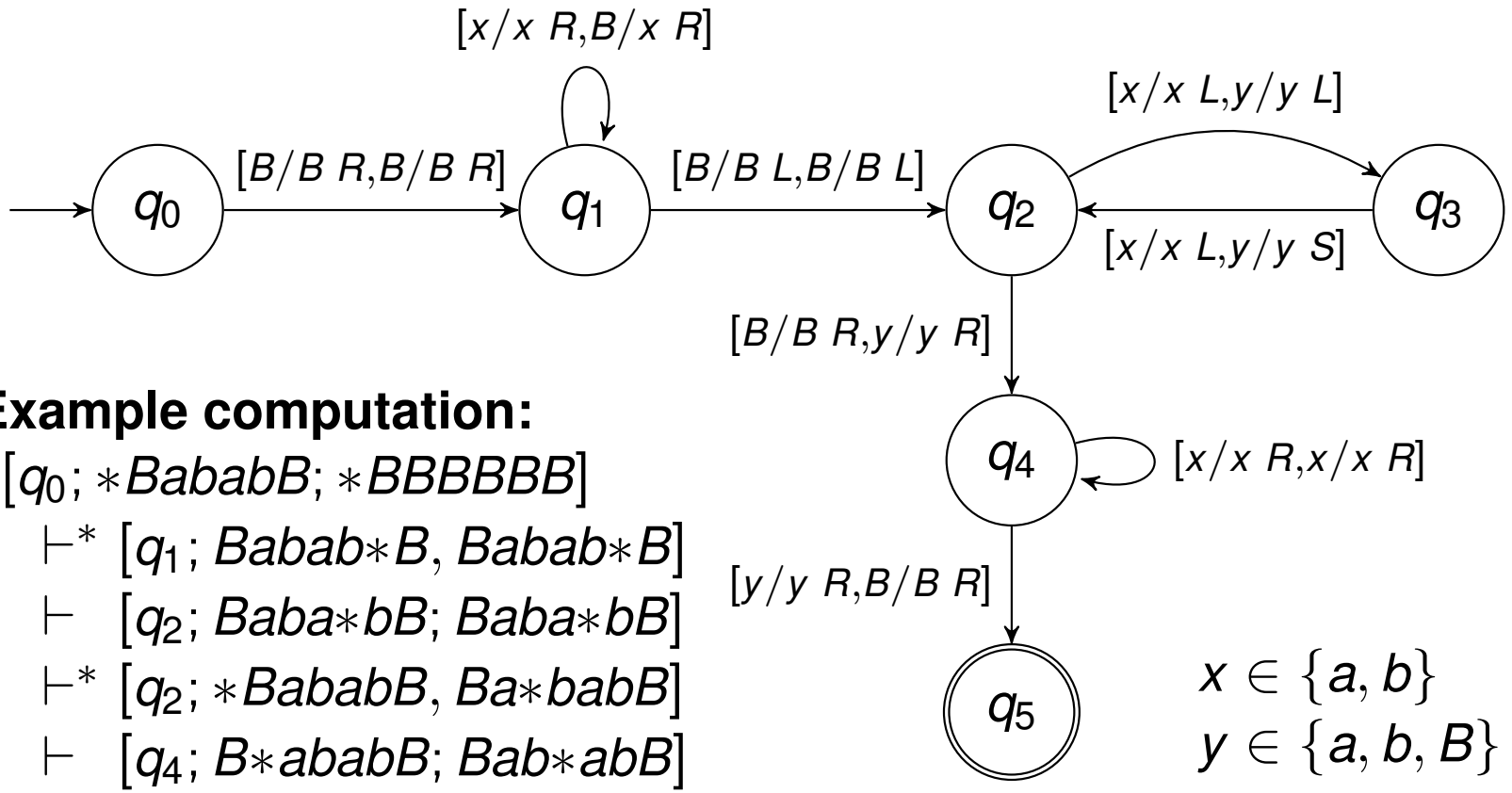

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Recognized language:

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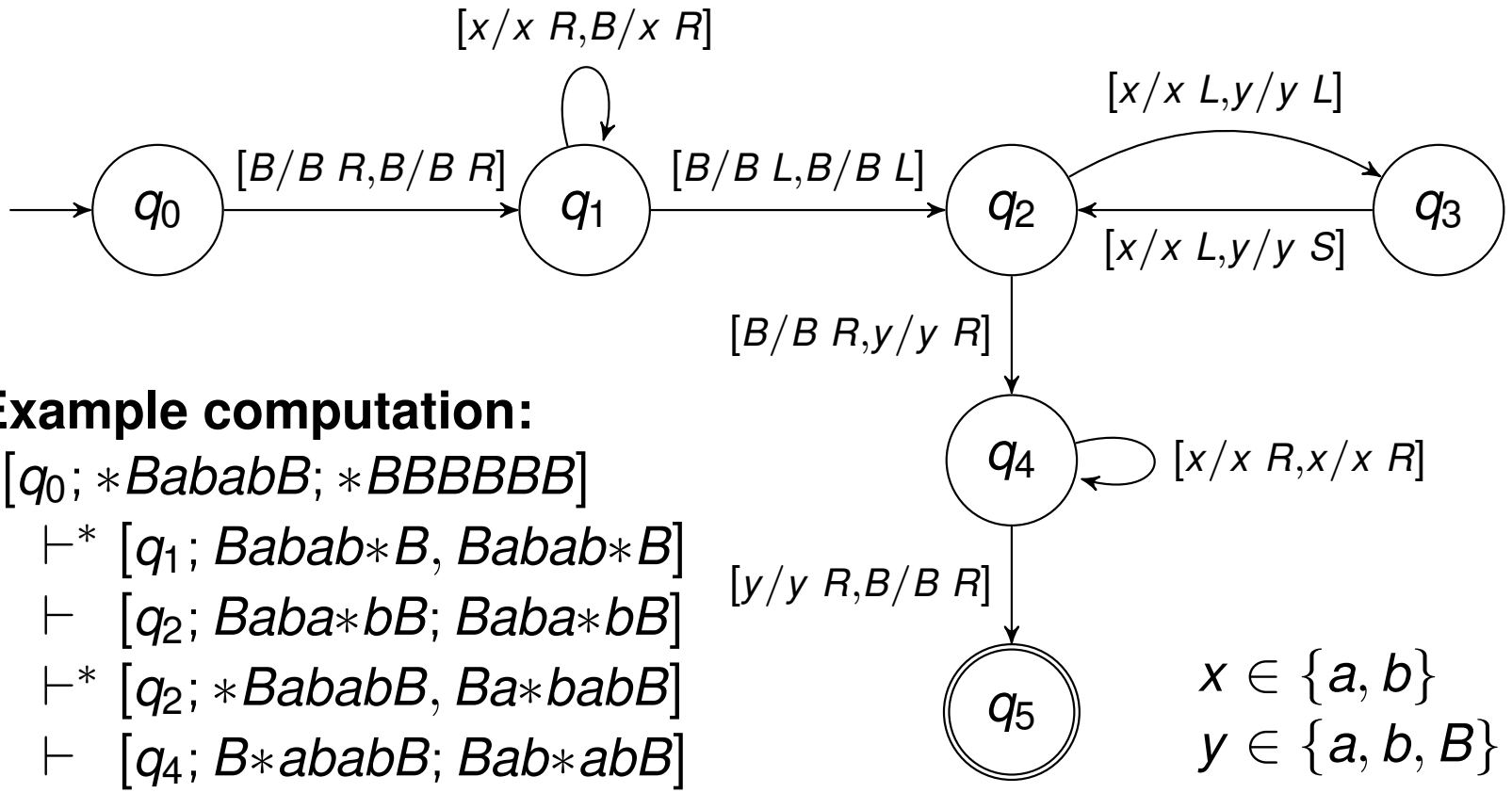

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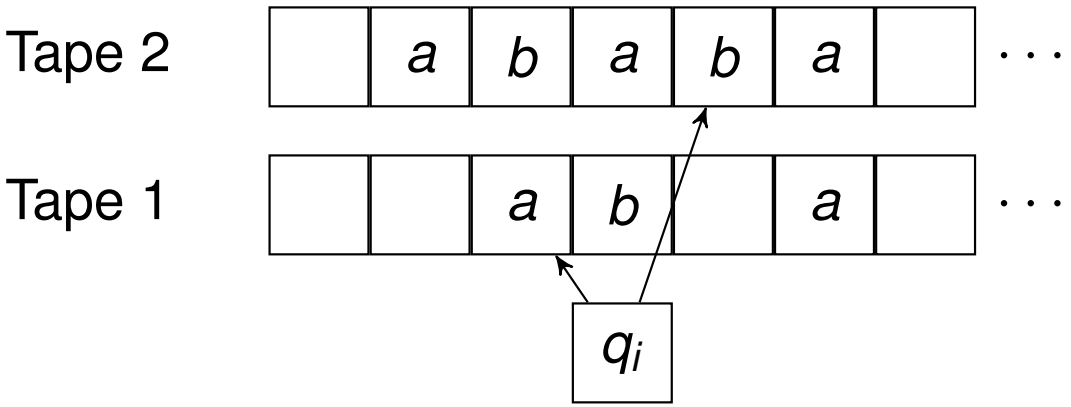
Recognized language: $\{ww \mid w \in \{a, b\}^*\}$

Proof idea of equivalence Multi-tape machine



is coded on one tape with five tracks:

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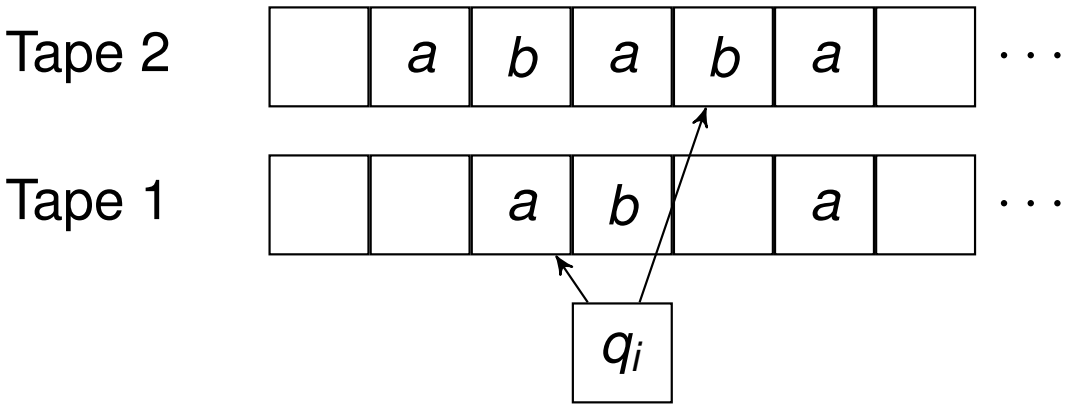

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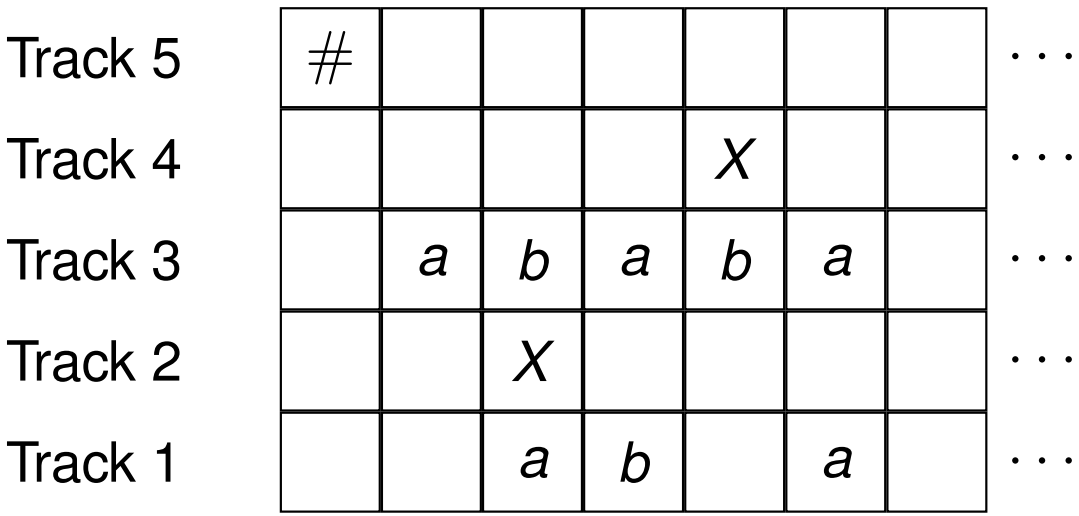

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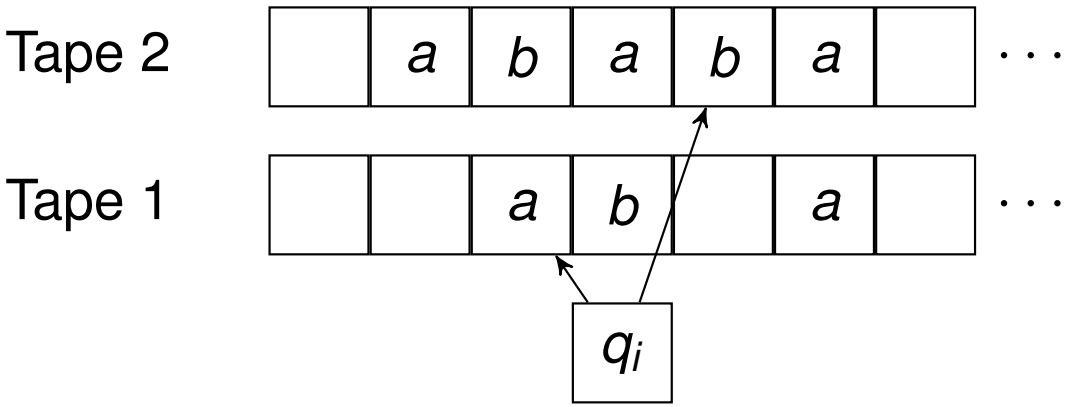

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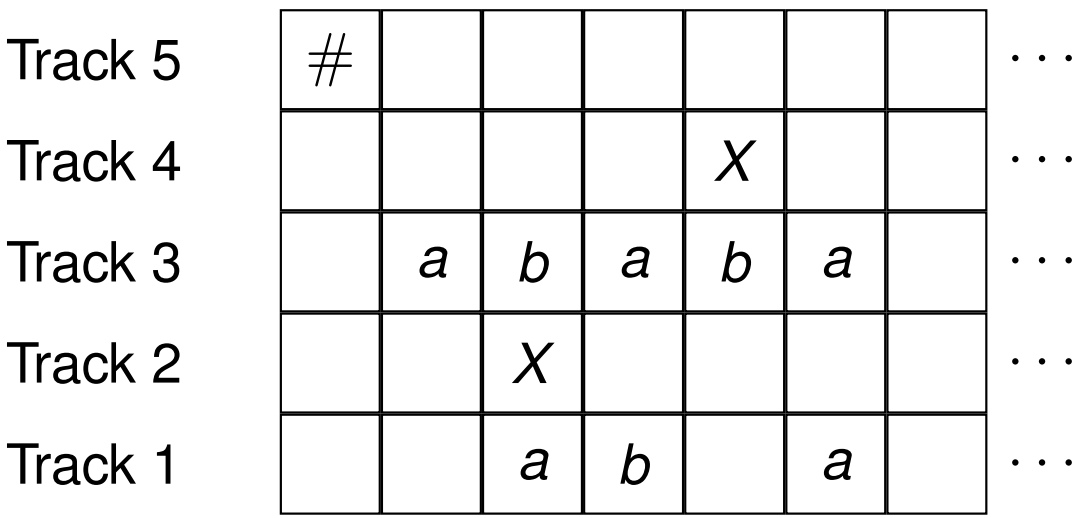

The functioning of the two-tape machine is now *simulated*.

Nondeterminism

Determinism: δ can be understood as a partial function

$$\delta : Q \times \Gamma \to \Gamma \times \{L, R\} \times Q$$

(Given q and x there exists at most one transition $q \xrightarrow{x/y\ d} q'$.)

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Does (non)determinism matter for expressiveness?

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A language is accepted by a nondeterministic machine if and only if it is recursively enumerable.

Expressiveness nondeterministic Turing machines

Proof idea: Backtracking

Suppose that L is accepted by nondeterministic TM M.

Now simulate M with a deterministic TM M' with 3 tapes:

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Suppose that L is accepted by nondeterministic TM M.
Now simulate M with a deterministic TM M' with 3 tapes:

- Tape 1: Remember the original input
- Tape 2: Record the current choices for backtracking $(m_1, \ldots, m_n)$
- Tape 3: Simulate n steps of M, according to choices on Tape 2

After the simulation of n steps: Replace Tape 2 by the following choice and restart Tape 3 with the contents of Tape 1.

Application: Recognize $(a+b)^i b(a+b)^i$: words with a b in the middle

Summary so far

- Turing machines
  - Memory is an infinite tape instead of just a stack (as in PDA))
  - Recursively enumerable languages (those accepted by a TM)
    - TM halts and accepts if $w \in L$ (yes/no/don't know)
  - Recursive languages (those decided by a TM)
    - TM decides the language, so it always halts (yes/no)
- Variations
  - Multi-tape Turing Machines
  - Nondeterministic Turing Machines

All variations recognise recursively enumerable languages!