Languages and Machines (Module 7 TCS+IAM)
L&M 6: Pushdown Automata (PDA) and Context-Free Languages
Chapter 7 (Sudkamp)

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Lecture 6

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Stack

Pushdown Automata extend Finite Automata with a Stack.
A Stack:

- holds an unbounded amount of elements
- can be accessed only in a restricted way
  - We can push an element on top of the stack
  - We can pop an element from the stack

LIFO = Last-in, First-out

Pushdown automata (PDAs)

Definition

A *pushdown automaton* is a tuple $\langle Q, \Sigma, \Gamma, \delta, q_0, F \rangle$

- Q is a set of states
- Σ is *input* alphabet
- Γ is *stack* alphabet
- $\delta \subseteq Q \times \Sigma^\lambda \times \Gamma^\lambda \times \Gamma^\lambda \times Q$ is transition relation ($X^\lambda = X \cup \{\lambda\}$)
- $q_0 \in Q$ is start state
- $F \subseteq Q$ is set of final states

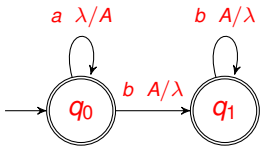
Differences with finite automata:

- Γ is the *stack alphabet* (cf. variables in CFGs)
- Transitions: 5-tuples (q, a, A, A', q') instead of 3-tuples (q, a, q')
- We write $q \xrightarrow{a \;\; A/A'} q'$ instead of $(q, a, A, A', q') \in \delta$

Computations of a pushdown automaton

- Configurations are combination $[q, w, \alpha]$, where $q \in Q$, $w \in \Sigma^*$, $\alpha \in \Gamma^*$ (α is the *stack*).
- Effect of a transition $q \xrightarrow{a \;\; A/B} q'$:
  - Input symbol a is *recognized* (can be λ)
  - Stack symbol A is *popped* (can be λ)
  - Stack symbol B is *pushed* (can be λ)
- Example:

$Q = \{q_0, q_1\}$
$\Sigma = \{a, b\}$
$\Gamma = \{A\}$
$F = \{q_0, q_1\}$
$\delta = \{(q_0, a, \lambda, A, q_0),$
$(q_0, b, A, \lambda, q_1),$
$(q_1, b, A, \lambda, q_1)\}$



Computation:
$[q_0, aabb, \lambda]$
$\vdash [q_0, abb, A]$
$\vdash [q_0, bb, AA]$
$\vdash [q_1, b, A]$
$\vdash [q_1, \lambda, \lambda]$

Done: $q_1 \in F$, stack is empty

- Language of this automaton? $\{a^i b^i \mid i \geq 0\}$

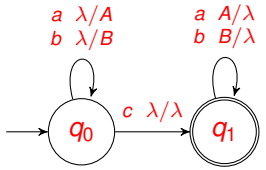
Language of a pushdown automaton

Definition

The language $L(M)$ of a pushdown automaton $M = \langle Q, \Sigma, \Gamma, \delta, q_0, F \rangle$ is the set of words w, for which a computation exists such that:

- beginning in $[q_0, w, \lambda]$
- ending in $[q, \lambda, \lambda]$ for certain $q \in F$.

Example: Language $\{wcw^R \mid w \in \{a, b\}^*\}$ (R the reverse operator)



$[q_0, abcba, \lambda]$
$\vdash [q_0, bcba, A]$
$\vdash [q_0, cba, BA]$
$\vdash [q_1, ba, BA]$
$\vdash [q_1, a, A]$
$\vdash [q_1, \lambda, \lambda]$

Connection with finite automata

- Both are *machine models*
- Pushdown automaton has unbounded (infinite) *memory*
- Memory is only partially accessible (only top of the stack)

Every finite automaton can be converted into a pushdown automaton

- Set of stack variables empty
- Stack always remains empty
- Regular transition $q \xrightarrow{a} q'$ converted to "stack transition" $q \xrightarrow{a \;\; \lambda/\lambda} q'$

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Deterministic pushdown automata

Definition: Overlapping transitions

Transitions $q \xrightarrow{x_1 \;\; A_1/B_1} q_1$ and $q \xrightarrow{x_2 \;\; A_2/B_2} q_2$ *overlap* if

- $x_1 = x_2$ or $x_1 = \lambda$ or $x_2 = \lambda$, ánd
- $A_1 = A_2$ or $A_1 = \lambda$ or $A_2 = \lambda$

Intuition: overlapping transitions leave options open

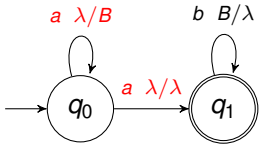
- $q \xrightarrow{a \;\; A/B_1} q_1$ versus $q \xrightarrow{a \;\; A/B_2} q_2$
- $q \xrightarrow{a \;\; A/B_1} q_1$ versus $q \xrightarrow{a \;\; \lambda/B_2} q_2$
- $q \xrightarrow{a \;\; A/B_1} q_1$ versus $q \xrightarrow{\lambda \;\; A/B_2} q_2$
- $q \xrightarrow{a \;\; \lambda/B_1} q_1$ versus $q \xrightarrow{\lambda \;\; A/B_2} q_2$
- $q \xrightarrow{a \;\; A/B_1} q_1$ versus $q \xrightarrow{\lambda \;\; \lambda/B_2} q_2$

Definition: Determinism

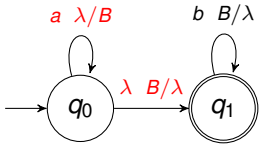
Pushdown automata without overlapping transitions are *deterministic*

Examples (non-)deterministic pushdown automata

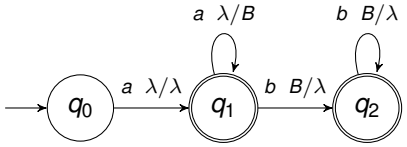
The following automata are equivalent (language: $\{a^m b^n \mid m = n + 1\}$)



overlap



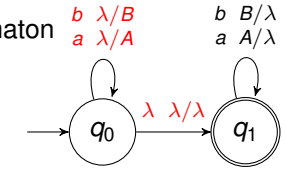
overlap



deterministic

Strength of deterministic pushdown automata

- Consider the automaton



- Language: $L = \{ww^R \mid w \in \{a, b\}^*\}$
- The outgoing transitions of q_0 overlap
  - When recognizing *abba* we should go to q_1 after *ab*
  - When recognizing *abbbba* we should remain in q_0 after *ab*
  - In general: the automaton must *guess* the middle of the word
- L has *no* deterministic pushdown automaton!
- Thus deterministic pushdown automata are less powerful
  - Different from finite automata

What's next?

Intriguing Matter

- We will find out if the following holds:
  $(L = \mathcal{L}(G)$ for a CFG $G) \Leftrightarrow (L = \mathcal{L}(M)$ for a PDA $M)$
- Does the following also hold?
  $L = \mathcal{L}(G)$ for an unambiguous CFG $G \Leftrightarrow$
  $L = \mathcal{L}(M)$ for a deterministic PDA M?
- No! Unambiguous grammar for ww^R: $S \to aSa \mid bSb \mid \lambda$
  Each PDA for palindromes must "guess" the middle.

The class of PDPAs is quite different from the class of PDAs

http://en.wikipedia.org/wiki/Deterministic_pushdown_automaton

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Playing with acceptance conditions

Definition: Acceptance by final state

The language $L_F(M)$ of a pushdown automaton $M = \langle Q, \Sigma, \Gamma, \delta, q_0, F\rangle$ is the set of words w with a computation,

- beginning in $[q_0, w, \lambda]$
- ending in $[q, \lambda, \alpha]$ with $q \in F$ and $\alpha \in \Gamma^*$.

Remark: The F in L_F stands for "final".

Definition: Acceptance by empty stack

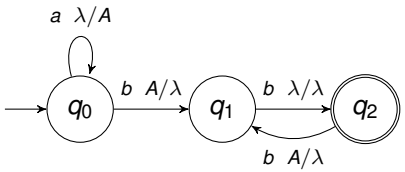
The language $L_E(M)$ of a pushdown automaton $M = \langle Q, \Sigma, \Gamma, \delta, q_0, F\rangle$ is the set of words w for which a *non-empty* computation exists, such that

- beginning in $[q_0, w, \lambda]$
- ending in $[q, \lambda, \lambda]$ for some $q \in Q$.

Remark: E in L_E stands for "empty". The F is now superfluous.

Alternative acceptance: Example

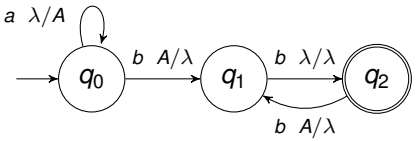
Example pushdown automaton:



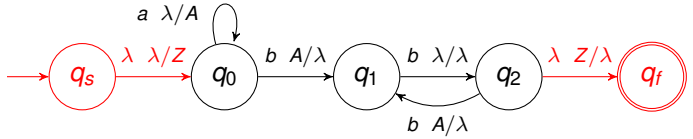
- $L(M) = \{a^n b^{2n} \mid n > 0\}$
- $L_F(M) = \{a^n b^{2k} \mid n > 0, 0 < k \leq n\}$
  ... (every computation that ends in q_2 is done)
- $L_E(M) = \{a^n(b^{2n-1} \cup b^{2n}) \mid n > 0\}$
  ... (the stack is already empty in q_1, so the computation is done)

From and to acceptance by final state: Example

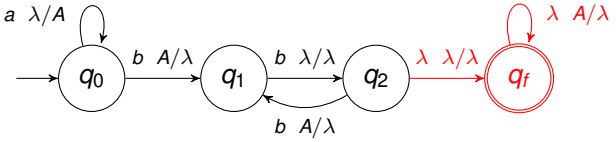
- $L(M) = \{a^n b^{2n} \mid n{>}0\}$ and $L_F(M) = \{a^n b^{2k} \mid n{>}0, 0{<}k{\leq}n\}$:



- M_1 such that $L_F(M_1) = L(M)$:



- M_2 such that $L(M_2) = L_F(M)$:



Acceptance by final state

Theorem: For every language $X \subseteq \Sigma^*$ the following is equivalent:

- A PDA M exists such that $L(M) = X$;
- A PDA M_F exists such that $L_F(M_F) = X$

Differently formulated:

Lemma: Transformation from and to acceptance by final state

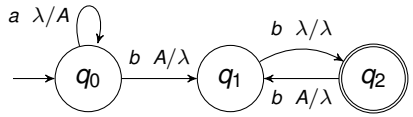
1. For every PDA M a PDA M' exists such that $L_F(M') = L(M)$
2. For every PDA M a PDA M' exists such that $L(M') = L_F(M)$

Construction

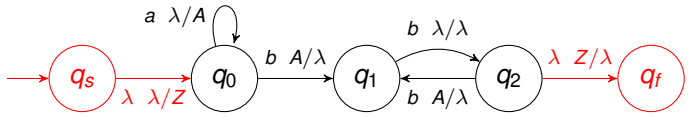
1. $Q' = Q \uplus \{q_s, q_f\}$, $\Gamma' = \Gamma \uplus \{Z\}$, $F' = \{q_f\}$;
   new transformations $q_s \xrightarrow{\lambda\ \lambda/Z} q_0$, and $q \xrightarrow{\lambda\ Z/\lambda} q_f$ (all $q \in F$)
2. $Q' = Q \uplus \{q_f\}$, $F' = \{q_f\}$; new transformations
   $q \xrightarrow{\lambda\ \lambda/\lambda} q_f$ (all $q \in F$), and $q_f \xrightarrow{\lambda\ A/\lambda} q_f$ (all $A \in \Gamma$)

From and to acceptance by empty stack: Example

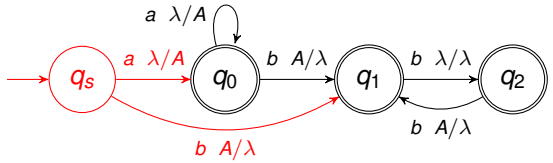
- M, with $L_E(M) = \{a^n(b^{2n-1} \cup b^{2n}) \mid n > 0\}$:



- M_3 such that $L_E(M_3) = L(M)$: Identical to M_1, except $F_3 = \emptyset$:



- M_4 such that $L(M_4) = L_E(M)$:



Acceptance by empty stack

Theorem: For every language $X \subseteq \Sigma^*$ the following is equivalent:

- A PDA M exists such that $L(M) = X$;
- A PDA M_E exists such that $L_E(M_E) = X$

This is equivalent to the following statements:

Lemma: Transformation from and to acceptance by empty stack

1. For every PDA M a PDA M' exists such that $L_E(M') = L(M)$
2. For every PDA M a PDA M' exists such that $L(M') = L_E(M)$

Construction

1. The same as for acceptance by final state!
2. $Q' = Q \uplus \{q_s\}$, $\Gamma' = \Gamma$, $F' = Q$; new transitions
   $q_s \xrightarrow{x\ A/B} q'$ for all $q_0 \xrightarrow{x\ A/B} q'$

Extended pushdown automata

Definition

An *extended* pushdown automata has a transition function

$$\delta \subseteq Q \times \Sigma^{\lambda} \times \Gamma^{\lambda} \times \Gamma^{*} \times Q$$

instead of

$$\delta \subseteq Q \times \Sigma^{\lambda} \times \Gamma^{\lambda} \times \Gamma^{\lambda} \times Q$$

and thus transitions of the form $q \xrightarrow{a \;\; A/W} q'$ with $W \in \Gamma^{*}$.

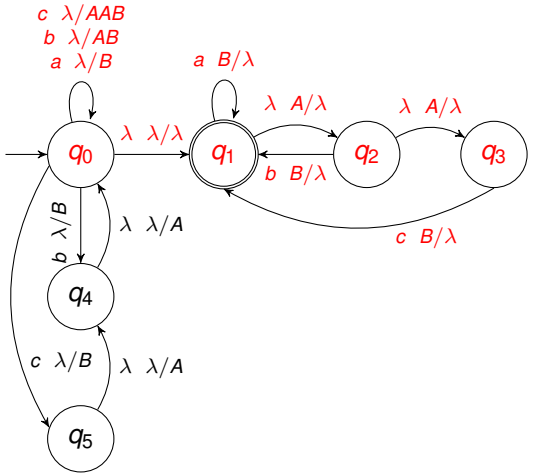
- Extended pushdown automata can push multiple symbols simultaneously
- Results in more compact automata

Transformation from extended to normal automata

An equivalent normal PDA exists for every extended PDA.

Extended pushdown automata: example

- Language: "even" palindromes over $\{a, b, c\}$



$[q_0, acbbca, \lambda]$
$\vdash [q_0, cbbca, B]$
$\vdash [q_0, bbca, AABB]$
$\vdash [q_0, bca, ABAABB]$
$\vdash [q_1, bca, ABAABB]$
$\vdash [q_2, bca, BAABB]$
$\vdash [q_1, ca, AABB]$
$\vdash [q_2, ca, ABB]$
$\vdash [q_3, ca, BB]$
$\vdash [q_1, a, B]$
$\vdash [q_1, \lambda, \lambda]$

- Equivalent normal pushdown automaton

Contents

From CFGs to PDAs

Definition: Greibach Normal Form (GNF)

$G = \langle V, \Sigma, P, S \rangle$ is in *Greibach normal form* if for all rules:

- $A \to a B_1 B_2 \cdots B_n$ ($n \geq 0$, all $B_i \neq S$), ór
- $S \to \lambda$

Theorem 1

An *extended* PDA M exists for every GNF G such that $L(M) = L(G)$ (even one with 2 states!)

Theorem 2

A GNF G' exists for every CFG G such that $L(G) = L(G')$ (continue transforming)

Theorem 3

A CFG G exists for every PDA M such that $L(M) = L(G)$ (it's possible, but a very tedious construction)

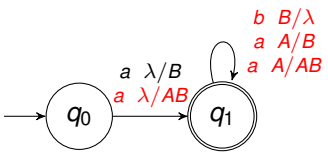
From CFGs to PDAs: Example

- GNF for $\{a^i b^i \mid i > 0\}$:

  $S \to aAB \mid aB$
  $A \to aAB \mid aB$
  $B \to b$

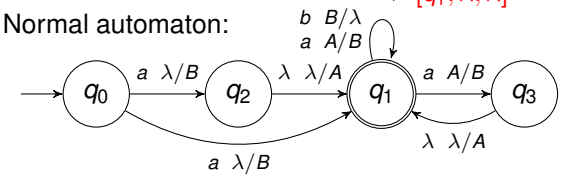
  $S \Rightarrow aAB \Rightarrow aaABB \Rightarrow aaaBBB \Rightarrow aaabBB \Rightarrow aaabbB \Rightarrow aaabbb$

- Constructed (extended) automaton:



$[q_0, aaabbb, \lambda]$
$\vdash [q_1, aabbb, AB]$
$\vdash [q_1, abbb, ABB]$
$\vdash [q_1, bbb, BBB]$
$\vdash [q_1, bb, BB]$
$\vdash [q_1, b, B]$
$\vdash [q_1, \lambda, \lambda]$

- Normal automaton:



From CFGs to PDAs (3)

Definition: Greibach Normal Form (GNF)

$G = \langle V, \Sigma, P, S \rangle$ is in *Greibach normal form* if for all rules:

- $A \to a B_1 B_2 \cdots B_n$ ($n \geq 0$, all $B_i \neq S$), ór
- $S \to \lambda$

Theorem

An *extended* PDA M exists for every GNF G such that $L(M) = L(G)$

Construction

$M = \langle \{q_0, q_1\}, \Sigma, V \setminus \{S\}, \delta, \{q_1\} \rangle$ with transitions

1. $q_0 \xrightarrow{\lambda \ \lambda/\lambda} q_1$ as $(S \to \lambda) \in P$;
2. $q_0 \xrightarrow{a \ \lambda/W} q_1$ for all $(S \to aW) \in P$, where $W \in V^*$;
3. $q_1 \xrightarrow{a \ A/W} q_1$ for all $(A \to aW) \in P$ and $A \in V \setminus \{S\}$

Construction GNF

Theorem

A GNF G' exists for every grammar G such that $L(G) = L(G')$

Transformation steps:

1. Stratify: order variables, and from low to high
   - Inlining rules that start with “lower” variable
   - Removing direct left recursion
2. From high to low: inlining remaining start variables
3. Introducing special non-terminals $R_a \to a$

Intermediate steps: Stratified grammar

G is *stratified* if the non-terminals are ordered, and

- $S \to \lambda$, or
- $A \to aw$ with $a \in \Sigma$ and $w \in (V \setminus \{S\} \cup \Sigma)^*$
- $A \to Bw$ with $B \in V$ and $w \in (V \setminus \{S\} \cup \Sigma)^*$, and $A < B$

Construction GNF: Example

Initial grammar: $S \to Aa$ $A \to Ac \mid Bc \mid b$ $B \to Aa$

1. Stratify
   - Order variables: $S < A < B$
   - Inline lower variables in S-rule: unnecessary
   - Inline lower variables in A-rule: unnecessary
   - Remove left recursion in A: Helper variable $A' < A$

     $A \to BcA' \mid bA' \mid Bc \mid b$
     $A' \to cA' \mid c$

   - Inline lower variable A in B-rule:

     $B \to BcA'a \mid bA'a \mid Bca \mid ba$

   - Remove left recursion in B: Helper variable $B' < B$

     $B \to bA'aB' \mid baB' \mid bA'a \mid ba$
     $B' \to cA'aB' \mid caB' \mid cA'a \mid ca$

Construction GNF: Example, steps 2 and 3

State of affairs: Stratified grammar

$$S \to Aa$$
$$A' \to cA' \mid c$$
$$A \to BcA' \mid bA' \mid Bc \mid b$$
$$B' \to cA'aB' \mid caB' \mid cA'a \mid ca$$
$$B \to bA'aB' \mid baB' \mid bA'a \mid ba$$

2. Inline higher variables (first $B > A$, then $A > S$)

$$A \to bA'aB'cA' \mid baB'cA' \mid bA'acA' \mid bacA' \mid bA' \mid$$
$$bA'aB'c \mid baB'c \mid bA'ac \mid bac \mid b$$
$$S \to bA'aB'cA'a \mid baB'cA'a \mid bA'acA'a \mid bacA'a \mid bA'a \mid$$
$$bA'aB'ca \mid baB'ca \mid bA'aca \mid baca \mid ba$$

3. Use special variables $R_a \to a$, $R_c \to c$

Overview: Seen so far

Pushdown automata (PDA's): $\langle Q, \Sigma, \Gamma, \delta, q_0, F \rangle$
- Stack alphabet Γ
- Transitions $q \xrightarrow{a \ A/B} q'$

Variations on pushdown automata:
- Deterministic pushdown automata: *less* powerful
- Extended transitions $q \xrightarrow{a \ A/W} q'$ with $W \in \Gamma^*$
- Acceptance by final state and/or empty stack

Connection between CFGs and PDAs
- Greibach normal form for CFGs
- Transformation CFGs $\to$ Greibach normal form
- Transformation Greibach normal form $\to$ extended PDAs
- Also possible: Transformation PDAs $\to$ CFGs

Next: Beyond Context-free languages
- Pumping lemma for CFGs
- Example: $\{a^i b^i c^i \mid i \geq 0\}$ not context-free
- Turing Machines and Unrestricted Grammars

Construction GNF: End result

We started with the following Context-free Grammer:

$$S \to Aa$$
$$A \to Ac \mid Bc \mid b$$
$$B \to Aa$$

We finally ended up with this equivalent Greibach Normal Form:

$$S \to bA'R_aB'R_cA'R_a \mid bR_aB'R_cA'R_a \mid bA'R_aR_cA'R_a \mid bR_aR_cA'R_a \mid bA'R_a \mid$$
$$bA'R_aB'R_cR_a \mid bR_aB'R_cR_a \mid bA'R_aR_cR_a \mid bR_aR_cR_a \mid bR_a$$
$$A \to bA'R_aB'R_cA' \mid bR_aB'R_cA' \mid bA'R_aR_cA' \mid bR_aR_cA' \mid bA' \mid$$
$$bA'R_aB'R_c \mid bR_aB'R_c \mid bA'R_aR_c \mid bR_aR_c \mid b$$
$$B \to bA'R_aB' \mid bR_aB' \mid bA'R_a \mid bR_a$$
$$A' \to cA' \mid c$$
$$B' \to cA'R_aB' \mid cR_aB' \mid cA'R_a \mid cR_a$$
$$R_a \to a$$
$$R_b \to b$$