Languages and Machines (Module 7 TCS+IAM)
L&M 2: Non-deterministic Finite Automata
Ch 5:4-6, Ch 6:2

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Lecture 2

Contents

1. Recap: deterministic finite automata (DFA)
2. Non-determinism (NFA)
3. λ-steps
4. From NFA to DFA
5. From RE to NFA
6. From NFA to RE
7. Conclusions

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Incomplete DFA

- Has a partial function: $\delta : Q \times \Sigma \rightharpoonup Q$

Acceptance by DFA through configurations ($\vdash$)

A configuration of a DFA is a pair $[q, w] \in Q \times \Sigma^*$

- q is the current state
- w is the word that is left to be read

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Accepted language

$$\mathcal{L}(M) = \{w \in \Sigma^* \mid M \text{ accepts the word } w\}$$

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Known already

- Regular languages are closed under union (by definition $E_1 \cup E_2$)
- DFA languages are closed under complement (complement accepting states)

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Solution

- Non-deterministic Finite Automata (NFA): Guess!

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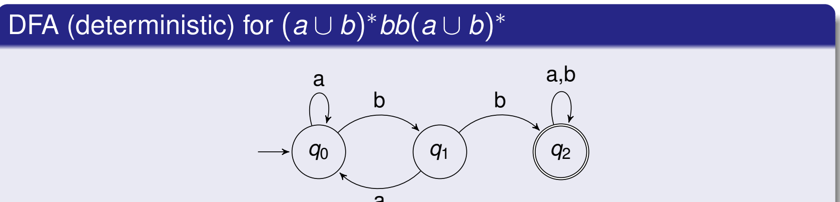

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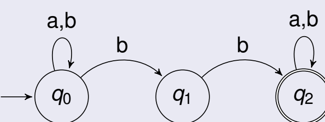
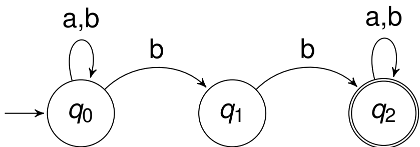


Non-determinism



There are four ways of tracing *bbb*

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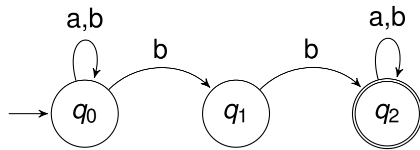

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$$
\begin{array}{ll}
 & [q0, bbb] \\
\vdash & [q0, bb] \\
\vdash & [q0, b] \\
\vdash & [q0, \lambda]
\end{array}
$$

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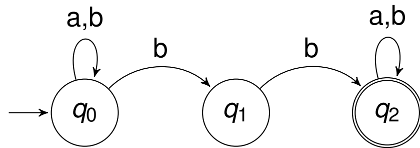

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	$[q0, bbb]$		$[q0, bbb]$
$\vdash$	$[q0, bb]$	$\vdash$	$[q0, bb]$
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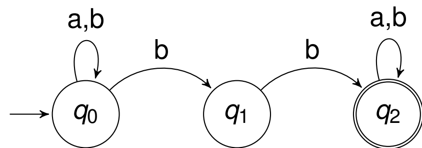

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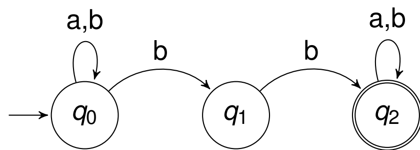

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$[q0, bbb]$	$[q0, bbb]$	$[q0, bbb]$	$[q0, bbb]$
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$\vdash [q0, b]$	$\vdash [q0, b]$	$\vdash [q1, b]$	$\vdash [q2, b]$
$\vdash [q0, \lambda]$	$\vdash [q1, \lambda]$	$\vdash [q2, \lambda]$	$\vdash [q2, \lambda]$

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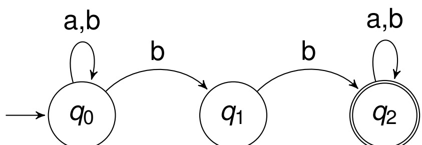

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$\vdash$	$[q0, bb]$	$\vdash$	$[q0, bb]$	$\vdash$	$[q0, bb]$	$\vdash$	$[q1, bb]$
$\vdash$	$[q0, b]$	$\vdash$	$[q0, b]$	$\vdash$	$[q1, b]$	$\vdash$	$[q2, b]$
$\vdash$	$[q0, \lambda]$	$\vdash$	$[q1, \lambda]$	$\vdash$	$[q2, \lambda]$	$\vdash$	$[q2, \lambda]$

How does this work exactly?

Is *bbb* now accepted??

Non-determinism (intuitive explanation)

When does an NFA accept a word?

- In a non-deterministic automation, a state can have multiple outgoing arrows with the same symbol.
- So, a word can correspond to multiple paths in the automaton. It might be the case that:
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- Definition: a word is accepted if there exists at least one path that leads to an accepting state
- (so non-deterministic automata always 'guess' right)

Guessing can be really helpful

string contains ‘ananas’ as DFA

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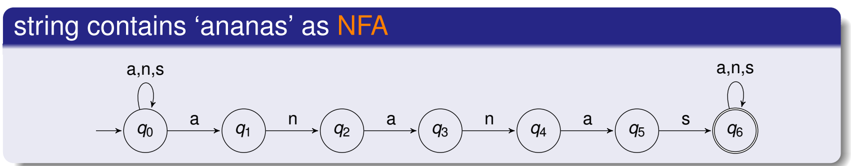
n,s
a
a
a
n
a,n,s
q_0 a q_1 n q_2 a q_3 n q_4 a q_5 s q_6
s
n,s
s
n,s



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A Non-deterministic Finite Automaton (NFA) is a 5-tuple

$$M = (Q, \Sigma, \delta, q_0, F)$$

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Nota bene

$\delta(q, a) \in \mathcal{P}(Q)$ is equivalent with $\delta(q, a) \subseteq Q$.

Every DFA “is” an NFA

Difference between DFA and NFA

- In a NFA, $\delta(q, a)$ is a set of states.
- This set can:
  - be empty (as in an incomplete DFA)
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  - contain multiple elements: real non-determinism

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- So the class of DFA-definable languages is contained in the class of NFA-definable languages. And the other way around??

Language accepted by an NFA

Acceptance through configurations

- Let $M = (Q, \Sigma, \delta, q_0, F)$ be an NFA.
- Define $[q, aw] \vdash [q', w]$ as $q' \in \delta(q, a)$
- M accepts w if there exists some computation of the form $[q_0, w] \vdash^* [q, \lambda]$ with $q \in F$.

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In other words

$$\mathcal{L}(M) = \{w \in \Sigma^* \mid \exists q \in F.\ [q_0, w] \vdash^* [q, \lambda]\}$$

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Helpful for the union of overlapping languages

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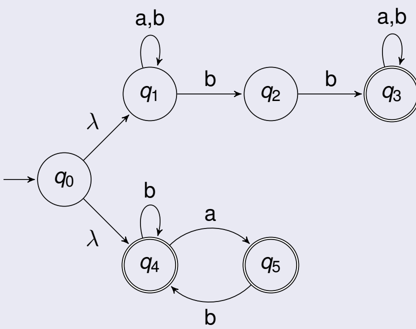

Automata with λ-transitions (NFA-λ)

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So automata with λ-transitions can either read a symbol,
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Semantics of NFA-λ (through λ-closure)

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- To describe all states q that can be reached with λ steps, we introduce λ-closure.

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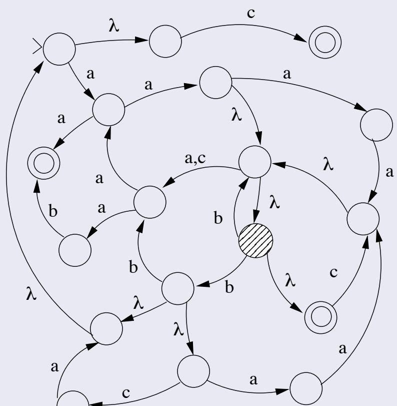
Inductive definition of λ-closure

- $q \in \lambda\text{-closure}(q)$.
- If $q_1 \in \lambda\text{-closure}(q)$, and $q_2 \in \delta(q_1, \lambda)$, then also $q_2 \in \lambda\text{-closure}(q)$.
- $\lambda\text{-closure}(q)$ is the smallest set that meets this requirement.

λ-closure (example)

illustration of λ-closure



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- What is the λ-closure after performing a b-step?
- Does this automaton accept b from the grey state?

λ-closure (example)

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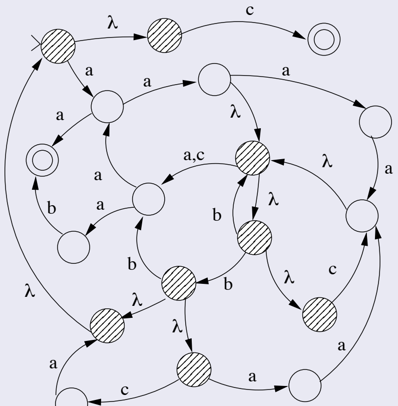

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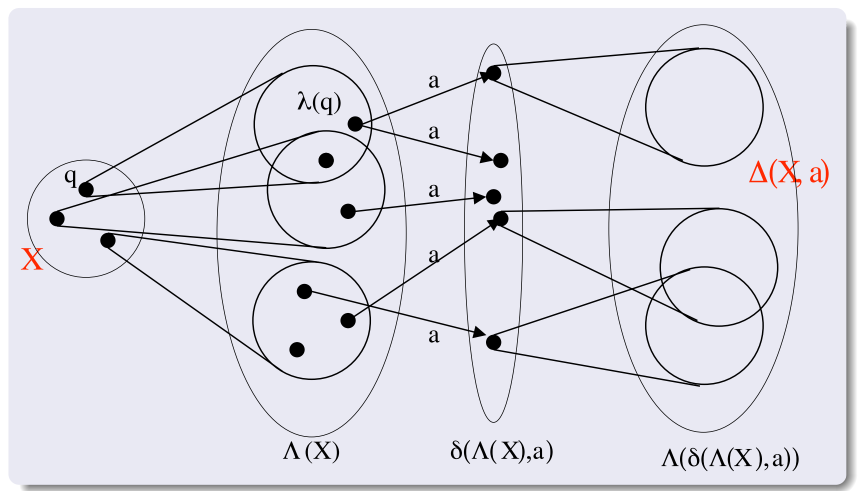
Impression of $\Delta(X, a)$



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  - Perform an a from those states and take the λ-closure again

$$\Delta(X, a) := \bigcup\{\Lambda(\delta(q, a)) \mid q \in \Lambda(X)\}$$

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- Applying δ on one state also produces a set
- We now extend λ and δ on sets as input.
- Suppose you are in a state of $X \subseteq Q$ and you perform an a:
  - First take the λ-closure of all elements of X:

$$\Lambda(X) := \bigcup \{\lambda\text{-closure}(q) \mid q \in X\}$$

  - Perform an a from those states and take the λ-closure again

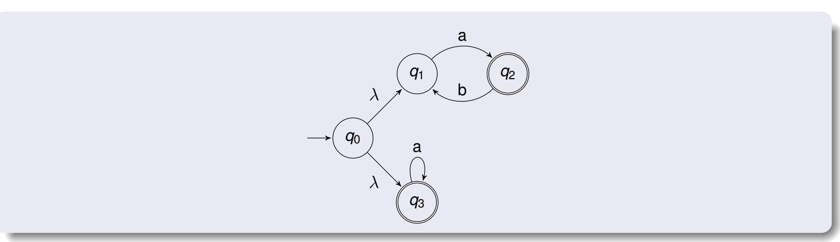
$$\Delta(X, a) := \bigcup \{\Lambda(\delta(q, a)) \mid q \in \Lambda(X)\}$$

Input Transition Function: $t : Q \times \Sigma \to \mathcal{P}(Q)$

Connection:

- $t(q, a) = \Delta(\{q\}, a)$
- $\Delta(X, a) = \bigcup \{t(q, a) \mid q \in X\}$

Example of input transition function



Transition function $t(q, a)$

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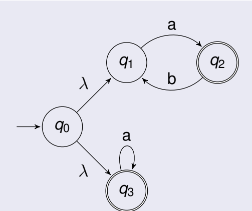

Transition function $t(q, a)$

	λ-closure	a	b
q_0	$\{q_0, q_1, q_3\}$	$\{q_2, q_3\}$	$\emptyset$

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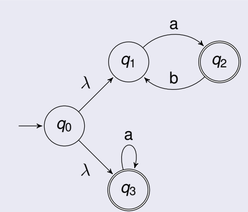

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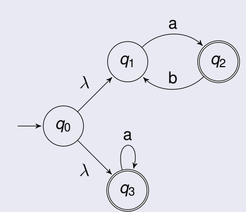

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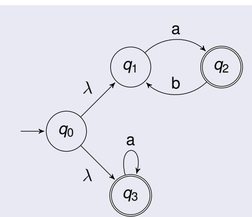

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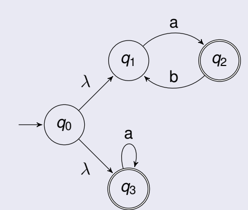

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$\Delta(\{q_1, q_3\}, a) = \{q_2, q_3\}$

From NFA-λ to DFA (1)

- Start with the NFA-λ: $M_N = (Q, \Sigma, \delta, q_0, F)$. We now have:

$$\Lambda(X) := \bigcup\{\lambda\text{-closure}(q) \mid q \in X\}$$

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- The construction is such that $\mathcal{L}(M_N) = \mathcal{L}(M_D)$. Conclusion:
  - for every NFA-λ, a DFA exists that accepts the same language
  - If the NFA contains n states, the DFA contains 2^n states

From NFA-λ to DFA: determinization

Idea to avoid "state space explosion" if this is possible

- Many of the 2^n states might be unreachable
- Alg. 5.6.3 in [Sudkamp]: only add states if necessary

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Algorithm

input: NFA-λ: $M_N = (Q, \Sigma, \delta, q_0, F)$
output: DFA $M_D = (Q', \Sigma, \Delta \cap (Q' \times Q'), \lambda\text{-closure}(q_0), F')$

$Q' := \{\lambda\text{-closure}(q_0)\}$; $U := Q'$;
while $(U \neq \varnothing)$ **do**
  choose X **from** U; $U := U - \{X\}$;

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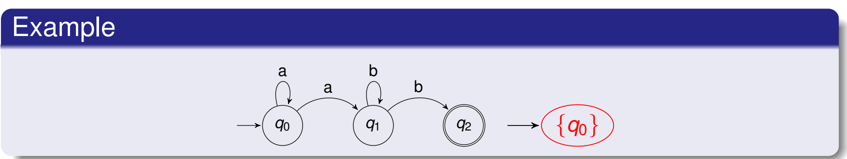
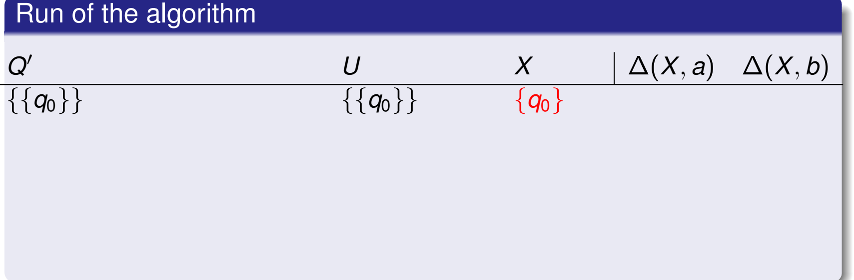
$F' := \{X \in Q' \mid X \cap F \neq \varnothing\}$;

Example (1) of determinization: a^+b^+

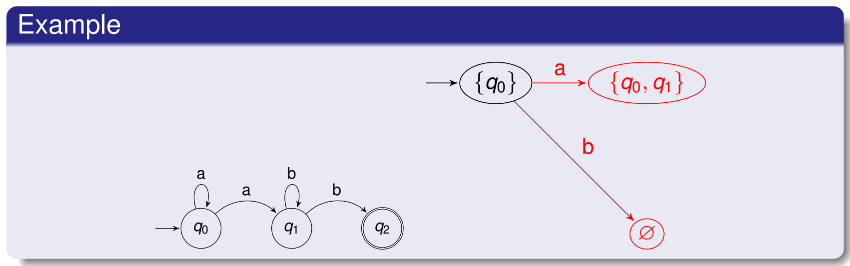




Q'	U	X	$\Delta(X,a)$	$\Delta(X,b)$
$\{\{q_0\}\}$	$\{\{q_0\}\}$	$\{q_0\}$		

Example (1) of determinization: a^+b^+

Example



Run of the algorithm

Q'	U	X	$\Delta(X, a)$	$\Delta(X, b)$
$\{\{q_0\}\}$	$\{\{q_0\}\}$	$\{q_0\}$	$\{q_0, q_1\}$	$\emptyset$

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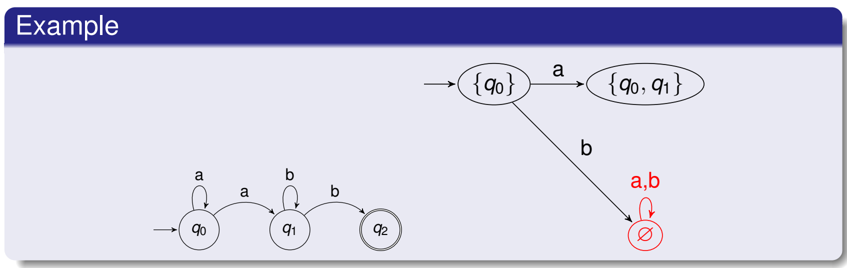
Example



Run of the algorithm

Q'	U	X	$\Delta(X, a)$	$\Delta(X, b)$
$\{\{q_0\}\}$	$\{\{q_0\}\}$	$\{q_0\}$	$\{q_0, q_1\}$	$\varnothing$
$\{\{q_0\}, \varnothing, \{q_0, q_1\}\}$	$\{\varnothing, \{q_0, q_1\}\}$	$\varnothing$	$\varnothing$	$\varnothing$

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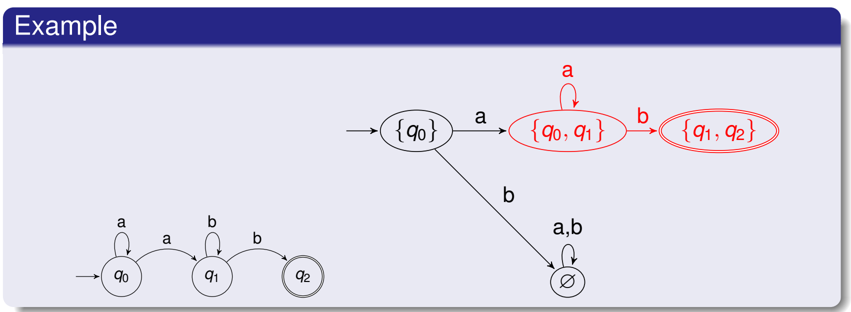

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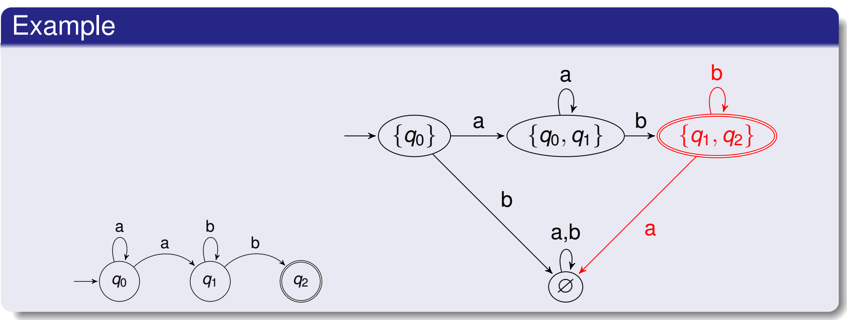

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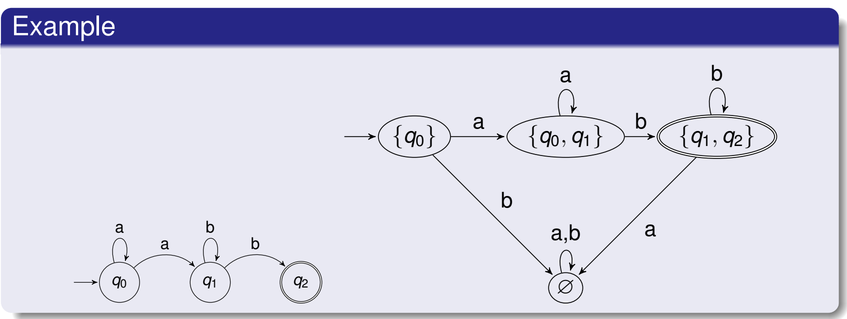

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Example (2) of determinization: with λ's

Input Transition Function t (almost as before)

	λ-closure	a	b
q_0	$\{q_0, q_1, q_3\}$	$\{q_2, q_3\}$	$\varnothing$
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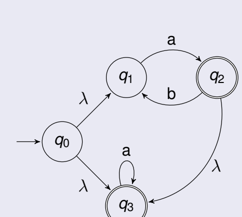

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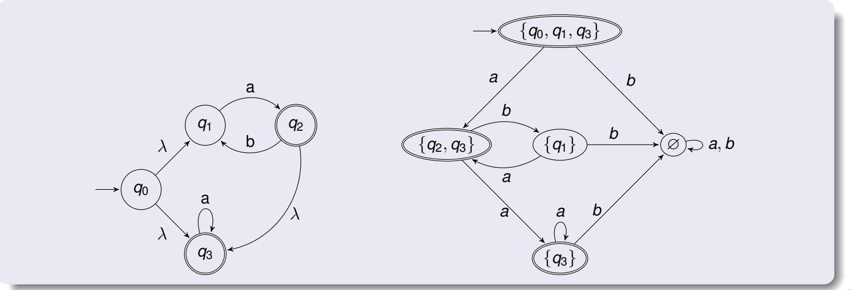

Contents

1. Recap: deterministic finite automata (DFA)
2. Non-determinism (NFA)
3. λ-steps
4. From NFA to DFA
5. From RE to NFA
6. From NFA to RE
7. Conclusions

Regular expressions (recap)

Regular expressions over Σ are defined inductively:

- Basis: $\emptyset$, λ and a (for $a \in \Sigma$) are regular expressions
- If E_1 and E_2 are regular expressions, then $(E_1 E_2)$ and $(E_1 \cup E_2)$ and (E_1^*) are also regular expressions
- There exist no other regular expressions

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- There exist no other regular expressions

A regular expression E defines a language $\mathcal{L}(E)$ as follows:

Expression E	Language $\mathcal{L}(E)$	Alternative notation
$\emptyset$	$\emptyset$	0
λ	$\{\lambda\}$	1 (of: ε)
a	$\{a\}$	a
$E_1 \cup E_2$	$\mathcal{L}(E_1) \cup \mathcal{L}(E_2)$	$E_1 + E_2$
$E_1 E_2$	$\mathcal{L}(E_1)\mathcal{L}(E_2)$	$E_1 \cdot E_2$
E^*	$\mathcal{L}(E)^*$	E^*

Composition of NFA-λ's

Example with λ's

- With NFA-λ we have sufficient ingredients to compose automata.

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- To simplify the composition, we work with composable NFA-λ, this means of the following form:
  1. The initial state q_0 has no incoming arrows (in-degree 0)
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  1. The initial state q_0 has no incoming arrows (in-degree 0)
  2. There is a unique final state
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- This can be visualized as:

- You can always make an NFA composable by introducing a new initial and final state, with some λ-steps

From Regular Expressions to NFA-λ

RE to NFA

- Theorem: For every regular expression E an equivalent NFA-λ exists, M, such that $\mathcal{L}(E) = \mathcal{L}(M)$.
- Proof: this can even be done with composable NFA-λ's. Induction over the construction of E.

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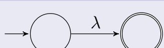
Base case: λ



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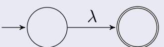

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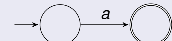

Regular Expressions to NFA-λ: Induction step

Induction step

- Suppose: we already have composable M_1 and M_2 for E_1 and E_2, with $\mathcal{L}(M_1) = \mathcal{L}(E_1)$ and $\mathcal{L}(M_2) = \mathcal{L}(E_2)$ (induction hypothesis).

Composable NFA-λ's for $E_1 E_2$, $E_1 \cup E_2$ and E_1^*

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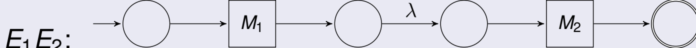
Composable NFA-λ's for $E_1 E_2$, $E_1 \cup E_2$ and E_1^*

$E_1 E_2$:



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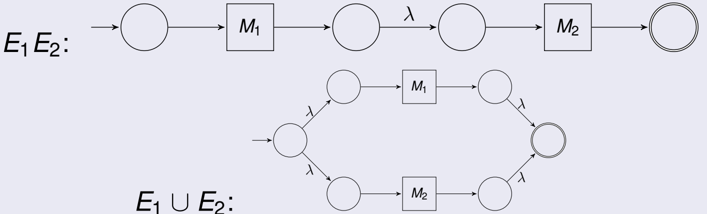

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$E_1 E_2$:

M_1 λ M_2

$E_1 \cup E_2$:

λ M_1 λ λ M_2 λ

E_1^*:

λ λ M_1 λ λ

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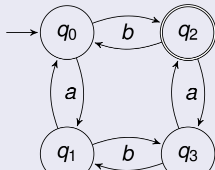
Example: even number of a's, odd number of b's

Complex example for regular expressions

- Try to write this as a regular expression (exercise *2.38 from the book)
- Write this as a DFA (example 5.3.7 from the book)
- Translate the DFA to a regular expression (exercise 6.3)

Reasonably simple DFA



From automata to regular expressions

The global idea

- Ensure that there is one accepting state (through extra λ's)
- Omit states step by step (elimination)
- This complicates the labels: regular expressions
- At the end, there are at most two states left:
  one initial and one accepting state; then retrieve the expression.

From automata to regular expressions

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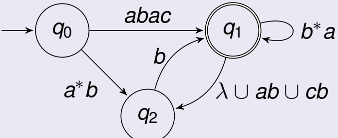
- Omit states step by step (elimination)
- This complicates the labels: regular expressions
- At the end, there are at most two states left:
  one initial and one accepting state; then retrieve the expression.

Expression graphs (= extended automata)

- NFA's with arbitrary regular expressions on the transitions.



- Note: Every NFA-λ is a special expression graph

Elimination of states (3 cases)

Before elimination	After elimination
If there is not a loop on q_i:	
$q_j \xrightarrow{E_1} q_i \xrightarrow{E_2} q_k$	$q_j \xrightarrow{E_1 E_2} q_k$

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Combining parallel transitions:	
$q_j \xrightarrow{E_1} q_k$ and $q_j \xrightarrow{E_2} q_k$	$q_j \xrightarrow{E_1 \cup E_2} q_k$

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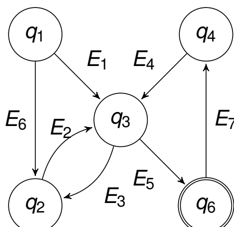

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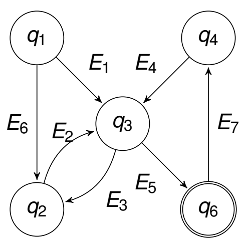

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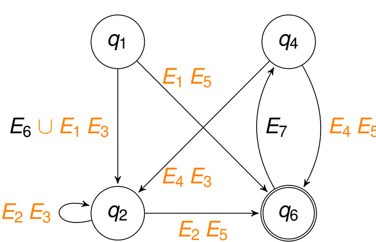

From Automata to Regular Expressions: final chord

- Expression graphs with 1 or 2 states can be directly converted into a regular expression. Some examples:

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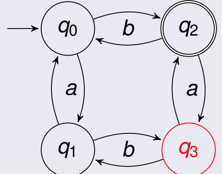
$\to q_0 \xrightarrow{E_2} q_1$ (final), loop E_3 on q_1	$E_2\, E_3^*$
$\to q_0$ with loop E_1, $q_0 \xrightarrow{E_2} q_1$ (final), loop E_3 on q_1, $q_1 \xrightarrow{E_4} q_0$	$E_1^* E_2 (E_3 \cup E_4 E_1^* E_2)^*$

Example: even number of a's, odd number of b's

Complex example

- Try to write as regular expression (exercise 2.38 from the book)
- Write this as a DFA (example 5.3.7 from the book)
- Translate the DFA to a regular expression (exercise 6.3)



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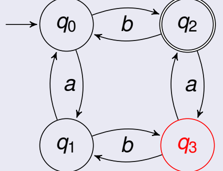

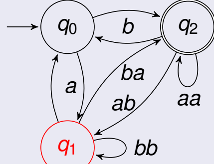

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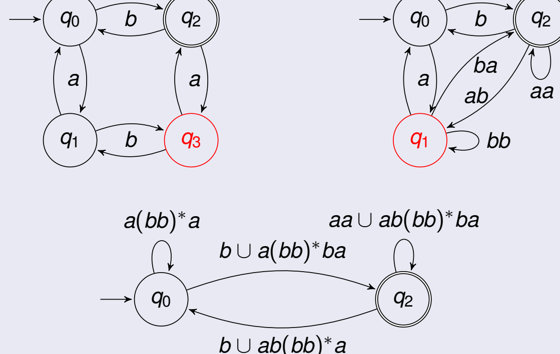

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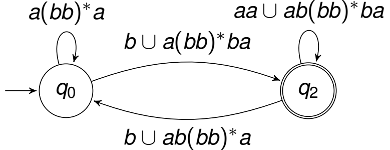

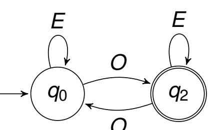

Retrieve the regular expression (with some use of imagination)

- Define $E := a(bb)^*a$ and $O = b \cup ab(bb)^*a$
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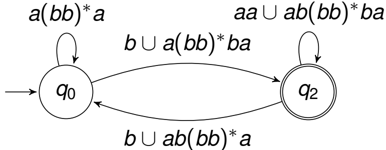

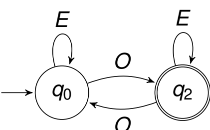

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- Then the requested expression is: $E^*O(E \cup OE^*O)^*$

Contents

1 Recap: deterministic finite automata (DFA)

2 Non-determinism (NFA)

3 λ-steps

4 From NFA to DFA

5 From RE to NFA

6 From NFA to RE

7 Conclusions

Conclusions so far: Regular Languages

- We define:
  - **REG** the class of regular languages (= definable with regexp)
  - **DFA** the class of DFA-definable languages
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  - **DFA** the class of DFA-definable languages
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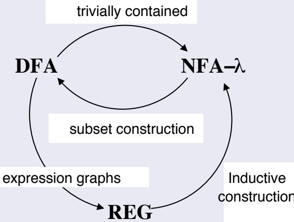
- The following diagram shows that:
  - The class **DFA** is contained in the class **REG**.
  - The class **REG** is contained in the class **NFA**.
  - So, the classes **DFA**, **NFA** and **REG** are all equal!



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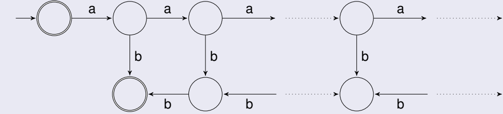

So what? .... Recursion in natural and artificial languages!

- Are the braces correct? ((((())(()()))(()))())
- Mathematical expressions, valid Python program, ...
- I beat the dog that gets the stick that is on the ground with another stick that it brought to me before.