Languages and Machines (Module 7 TCS+IAM)

L&M 1: Languages, Regular Expressions, Automata

Ch 1 (preliminaries); Ch 2:1-4; Ch 5:1-3

Alexander Skopalik

Discrete Mathematics and Mathematical Programming, Applied Mathematics, University of Twente

Lecture 1

Contents

Contents

Logistics (1)

Teachers

- Lectures: Alexander Skopalik
- Exercises: Marcus Gerhold, Alexander Skopalik, Carlos Budde

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Mandatory materials

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  difficult to get, alternative:
- *Introduction to Automata Theory, Languages, and Computation* , Hopcroft, John E.; Motwani, Rajeev; Ullman, Jeffrey D. , 3rd edition. Main difference: $\lambda \rightarrow \epsilon$.
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Optional tool

- JFLAP can be used to demonstrate most of our constructions.

Logistics (2)

Setup of theme Languages and Machines:

- Lecture: discussion of the course subjects, incomplete!
- Make sure to read all material from the book!
- Goal of the exercises: proofs and constructions – exam training
- Bring your book to the exercises!
- Bonus points: 12/3 and 26/3
- Use JFLAP on bigger examples, increase your understanding
- Link to project: Partition Refinement = minimization of automata
- Exam: March 27 (08:45), closed book

Goal of “Languages and Machines”

Mathematical studies of “computation”

- What is computing?
  - Transformation from input to output
  - input: sequence of symbols = word
  - output: sequence of symbols, or one decision (1 bit yes/no)

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What do we wish to accomplish?

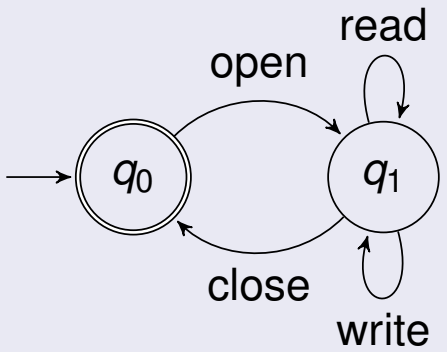
- Finite descriptions of infinite sets of words = language
  - Automata, Expressions, Grammars, Turing Machines
- Algorithms to manipulate those descriptions = compilers
  - transformations, analyzing, compilers
- Prove limits to what is computable in principle

Concrete Example: a correct file system

Regular Expression

$$(open\ (read + write)^*\ close)^*$$

Finite Automaton



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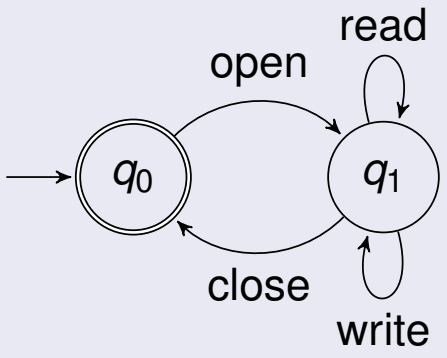

Some correct sequences:

open, read, read, write, close;
open, close, open, close;
open, read, close, open, read, read, close;

Some incorrect sequences:

open, open;
open, read, write;
open, read, write, close, write;
read, write;

We are interested in the Language of words that are accepted by this Regular Expression / Finite Automaton.

Typical theoretical questions

1. Can we define the same languages by using Regular Expressions and by using Finite Automata?
2. If so, how can we transform any Regular Expression into a Finite Automaton? (and vice versa)
3. How can we check (by an algorithm) that two given Finite Automata accept the same language?
4. Can we compute the smallest Finite Automaton that accepts some language?
5. Are there languages that cannot be expressed by Finite Automata at all? Can this be proved rigorously?
6. Are there stronger mechanisms than RE / FA to define languages?
7. Is there a universal mechanism that can define all languages?

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Applications on language . . . . . hero: Noam Chomsky (1928)

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- Compiler construction: definition/analysis of progr. languages
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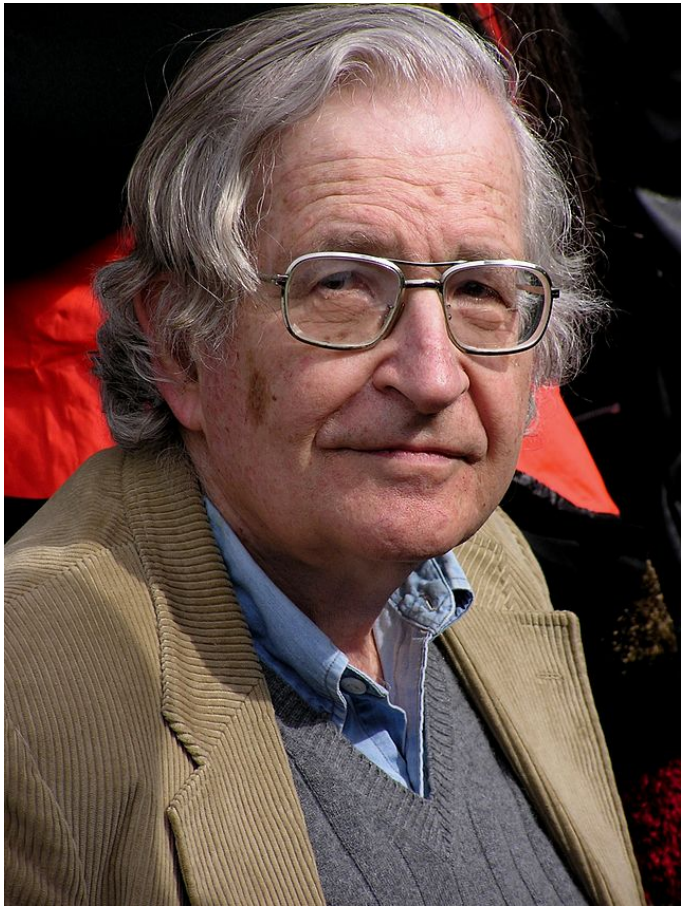
Calculators with increasing computing power . . . . . hero: Alan Turing

- Automata with finite memory . . . . . (starting today)
- Automata with infinite memory . . . . . (2nd half of lectures)

Heroes and pioneers



Noam Chomsky (1928)

Alan Turing (1912-1954)

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Somewhat more formal:

1. Given an alphabet Σ.
2. Σ^* is the set of all words $a_1 \dots a_n$,
   such that $a_i \in \Sigma$ for every $1 \leq i \leq n$ ($n = 0$ is possible).
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Examples from natural language

The language of English words: (in principle finite)

1. Alphabet: letters a-z,
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The language of English sentences: (in principle infinite)

1. Alphabet: words (day, bites, dog, the, ...)
2. Words: sequences of words, e.g.: "The The The", "The dog bites"
3. Language: The set of grammatically correct sentences
   - yes: "the dog bites the cat", "the day bites the day", "the dog bites the cat, that catches the mouse"
   - no: "the dog cat bites day"

Technical examples

Programming languages

- Alphabet: keywords, identifiers, etc.
- Language: all correct expressions in Matlab
- Language: alle correct Java programs

System specifications

- Alphabet: events/actions (input/output)
- e.g. elevator: {up,down,open,closed}
- e.g. rails: barriers open/closed, switch left/right, signal green/yellow/red
- Language: safe behavior = set of specific sequences of events

Abstract examples

$\Sigma = \{ (,) \}$, language consists of "correctly spelled" words

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- yes: $1 + 3$, $(13 * 24) + 18$, ...
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How to exactly define these languages?

- above languages contain infinitely many words.
- every separate word has finite length.

Operations on words (induction, recursion)

Recursive definition Σ^* (set of words, eq. 2.1.1)

1. λ is the empty word
2. If $a \in \Sigma$ and w is already a word, then aw is also a word
3. All words can be obtained by applying (1) and (2) finitely often.

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Recursive defintion $u \cdot v$ (concatenation, eq. 2.1.3)

Given u and v, define word $u \cdot v$ recursive (over u):

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Proof with induction over u that $(u \cdot v) \cdot w = u \cdot (v \cdot w)$

More operations on words:

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- Concatenation: uv (from here on we omit $\cdot$)

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Reverse (the reverse word)

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- Theorems: $(uv)^R = v^R u^R$ (proof with induction over u!)
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If $u \in \Sigma^*$ and $n \in \mathbb{N}$, define u^n recursive over n:

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Proof by yourself: $u^m u^n = u^{m+n}$ and $(u^m)^n = u^{mn}$

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Set operations on languages

- Union: $L_1 \cup L_2 = \{u \mid u \in L_1 \vee u \in L_2\}$
- Intersection: $L_1 \cap L_2 = \{u \mid u \in L_1 \wedge u \in L_2\}$
- Complement: $\overline{L} = \{u \mid u \in \Sigma^* \wedge u \notin L\}$

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Theorem: $|L_1 L_2| \leq |L_1||L_2|$ and $|L_1 \cup L_2| \leq |L_1| + |L_2|$

Repeat-operations on languages

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Two important sets (Kleene star)

$$L^* = \bigcup_{i=0}^{\infty} L^i = L^0 \cup L^1 \cup L^2 \cup \cdots \text{ 0 times or more}$$

$$L^+ = \bigcup_{i=1}^{\infty} L^i = L^1 \cup L^2 \cup L^3 \cup \cdots \text{ 1 time or more}$$

Contents

More about Kleene Star
after: Stephen Cole Kleene (1909-1994)

Observations

- $L^+ = LL^*$
- $L^* = \{\lambda\} \cup L^+$
- If $\lambda \in L$, then $L^* = L^+$

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  (see also [Sudkamp] table 2.1)

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- $\{a, b\}^*$ contains λ, a, b, $aaabb$, $a^{10}b^{12}$, a^{199}, $ab^{100}a$
- $\{ab, ba\}^+$ contains ab, $baba$, $abbaabba$, but not $abbaaba$ or λ
- $(\{a, b\}^* \cup \{cc\}^*)^*$ contains $\underline{aab}\,\underline{cc}\,\underline{cc}\,\underline{aa}\,\underline{ba}\,\underline{cccc}$

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- Question: does $(\{a, b\}^* \cup \{cc\}^*)^* = (\{a, b\} \cup \{cc\})^*$ hold?

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- $(\{a, b\}^* \cup \{cc\}^*)^*$ contains $\underline{aab}\,\underline{cc}\,\underline{cc}\,\underline{aa}\,\underline{ba}\,\underline{cccc}$
- Question: does $(\{a, b\}^* \cup \{cc\}^*)^* = (\{a, b\} \cup \{cc\})^*$ hold?
- Question: does $(\{a, b\} \cup \{cc\})^* = (\{a, b\} \cup \{c\})^*$ hold?

Regular expressions (important!)

Regular expressions over Σ are defined inductively:

- Basis: $\varnothing$, λ and a (for $a \in \Sigma$) are regular expressions
- If E_1 and E_2 are regular expressions, then $(E_1 E_2)$, $(E_1 \cup E_2)$ and (E_1^*) are also regular expressions
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A regular expression E defines a languages $\mathcal{L}(E)$ as follows:

Expression E	Language $\mathcal{L}(E)$	Alternative notation
$\varnothing$	$\varnothing$	0
λ	$\{\lambda\}$	1 (or: ε)
a	$\{a\}$	a
$E_1 \cup E_2$	$\mathcal{L}(E_1) \cup \mathcal{L}(E_2)$	$E_1 + E_2$
$E_1 E_2$	$\mathcal{L}(E_1)\mathcal{L}(E_2)$	$E_1 \cdot E_2$
E^*	$\mathcal{L}(E)^*$	E^*

Regular languages

L is regular, if a regular expression E exists, such that $L = \mathcal{L}(E)$

- A regular language over Σ is thus a language that can be described by a regular expression.
- Not every language can be defined with a regular expression (but how to prove this in concrete instances?) (discussed later)

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  I.e.: if L_1 and L_2 are regular, then is $L_1 \cap L_2$ regular?
  - Yes, but how can you see that??
  - Given E_1 and E_2. How to systematically construct E_3, such that $\mathcal{L}(E_3) = \mathcal{L}(E_1) \cap \mathcal{L}(E_2)$?
  - For example: $(aaa)^* \cap (aa)^* = (aaaaaa)^*$

Examples of regular languages (pattern matching)

Examples, where $\Sigma = \{a, b, c\}$

- All words that start with *ab* and end with *ba*
  - mathematically: $\{abwba \mid w \in \Sigma^*\} \cup \{aba\}$
  - regular expression:

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- All words that do not contain *ab* (difficult!): $(b \cup a^* c)^* a^*$

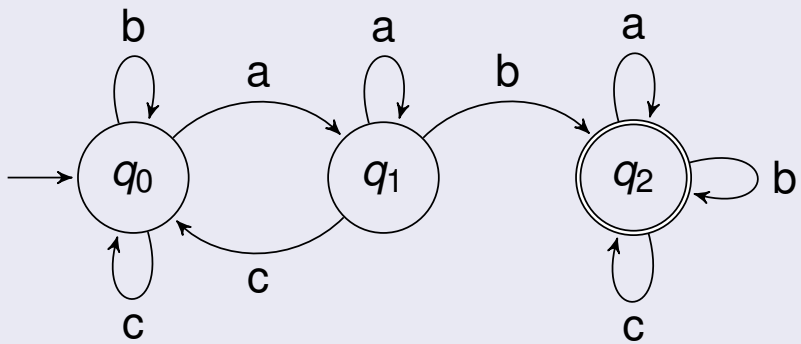
More open problems:

- How to compute the complement of E systematically?
- How to prove that $\{a^n b^n \mid n \in \mathbb{N}\}$ is not regular?

Contents

Deterministic finite automata (examples)

Language of words over $\{a, b, c\}$ that contain ab:



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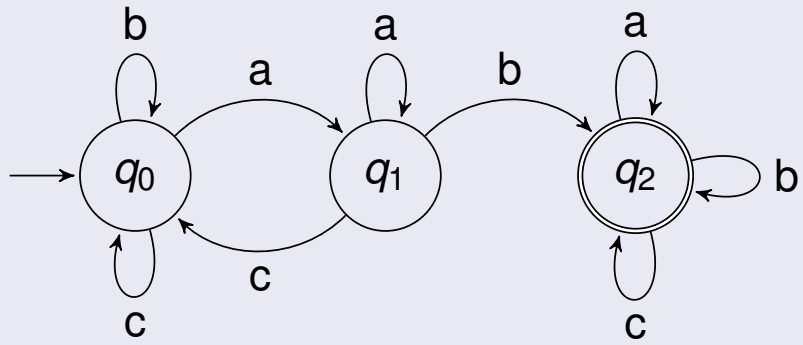

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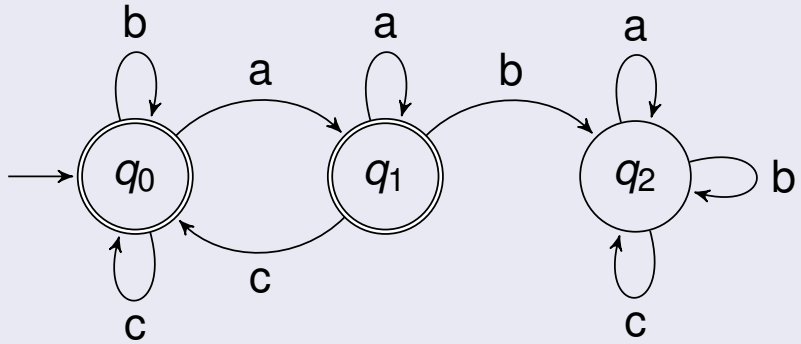

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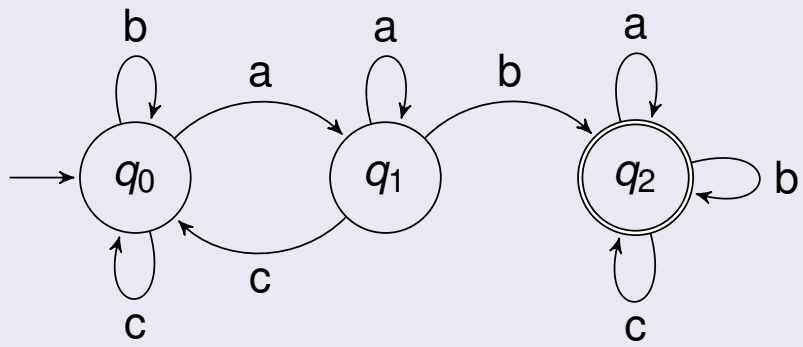

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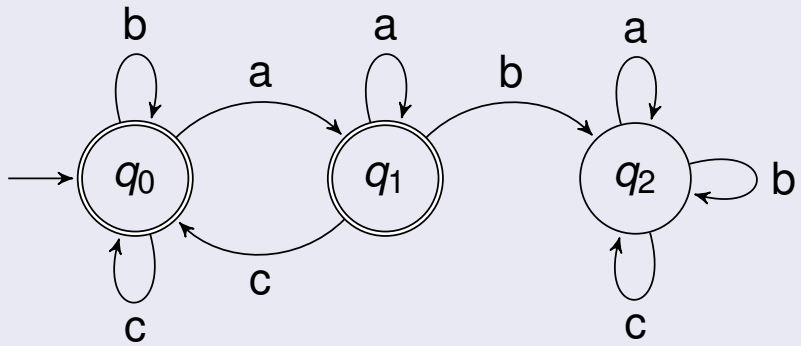


- apparently the complement is simple for a DFA

Deterministic finite automaton (definition)

A Deterministic Finite Automaton (DFA) is a 5-tuple:

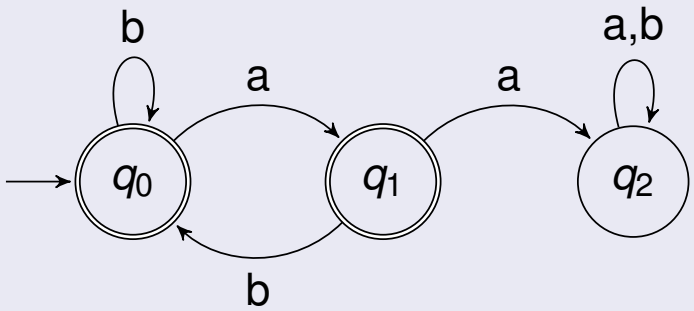
$$M = (Q, \Sigma, \delta, q_0, F)$$

Here, the components are:

- Q is a finite set of states
- $q_0 \in Q$ is the unique initial state
- Σ is a finite set of symbols, the alphabet.
- δ is a total function, $\delta : Q \times \Sigma \to Q$ (transition function)
- $F \subseteq Q$ is the set of accepting states.

Three equivalent notations

State diagram:



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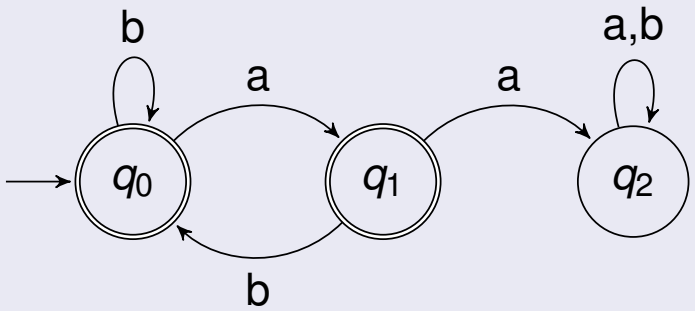

Mathematical definition:

$Q = \{q_0, q_1, q_2\}; \Sigma = \{a, b\}; F = \{q_0, q_1\};$
$\delta(q_0, a) = q_1, \delta(q_1, a) = q_2, \delta(q_2, a) = q_2,$
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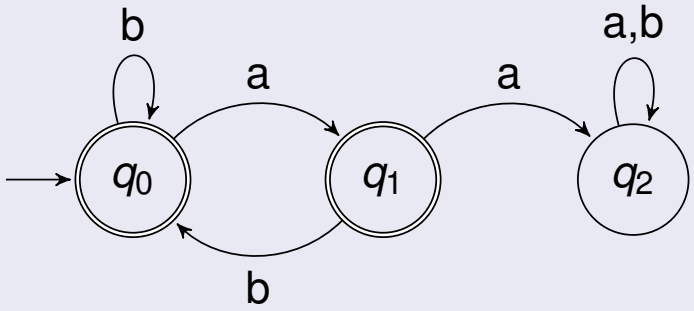

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Transition table for δ:

δ	a	b
q_0	q_1	q_0
q_1	q_2	q_0
q_2	q_2	q_2

Language accepted by a finite automaton (DFA)

the language that M accepts

Let $M = (Q, \Sigma, \delta, q_0, F)$,

$$\mathcal{L}(M) = \{w \in \Sigma^* \mid M \text{ accepts the word } w\}$$

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the language that M accepts

Let $M = (Q, \Sigma, \delta, q_0, F)$,

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Simple intuition (will later be formally defined)

- You start in the initial state q_0
- You read the word w symbol by symbol, and continuously follow the corresponding arrows according to δ
- At the end, you check whether you are in an accepting state of F

Acceptation of words (exactly, via $\vdash$)

A configuration of a DFA is a pair $[q, w] \in Q \times \Sigma^*$

- q is the current state
- w is the word that is left to be read

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Let $M = (Q, \Sigma, \delta, q_0, F)$ and define sequentially

- A one-step-relation $\vdash \subseteq (Q \times \Sigma^*) \times (Q \times \Sigma^*)$:

$$[q_i, aw] \vdash [\delta(q_i, a), w] \quad \text{(read: “infers”)}$$

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- The reflexive-transitive closure $\vdash^*$:
  - Contains $\vdash$: If $[q, w] \vdash [q', w']$ then $[q, w] \vdash^* [q', w']$
  - Reflexive: $[q, w] \vdash^* [q, w]$
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- M accepts w, if a $q \in F$ exists, with $[q_0, w] \vdash^* [q, \lambda]$.

Acceptance of words (exactly, via $\hat{\delta}$)

The extended transition function has type $\hat{\delta} : Q \times \Sigma^* \to Q$

- Let $M = (Q, \Sigma, \delta, q_0, F)$.
- Define $\hat{\delta}(q, u)$ recursively over u:
  - Basis: $\hat{\delta}(q, \lambda) = q$
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Theorem: these two definitions are equivalent.

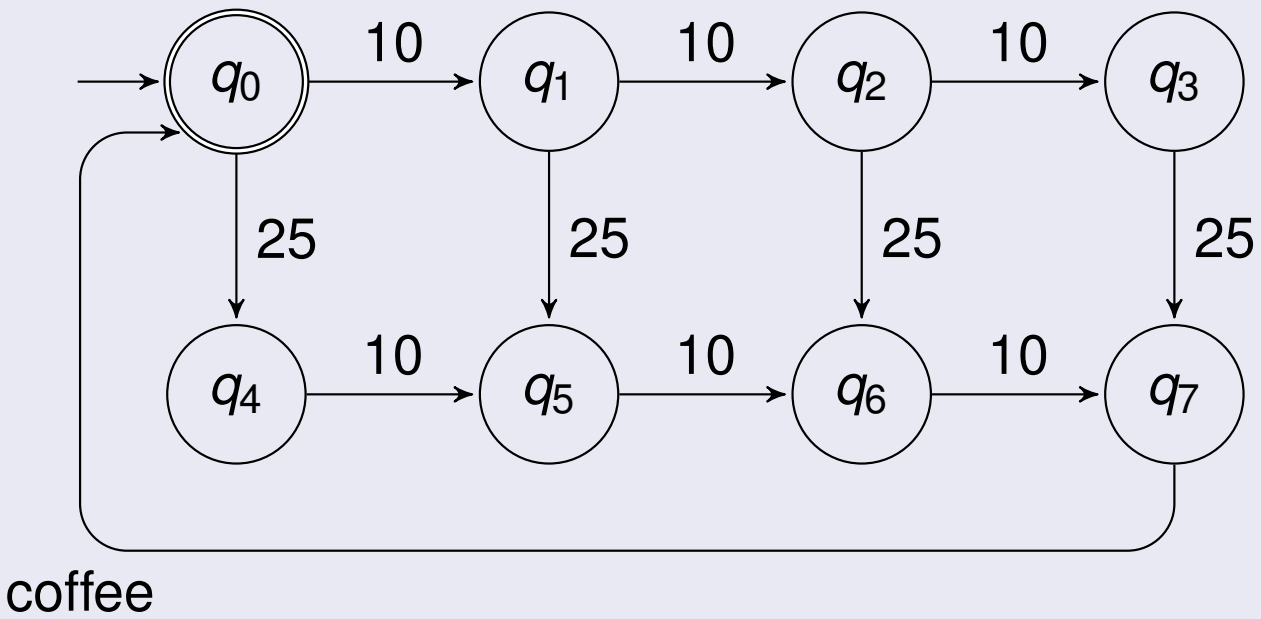
- In particular:
  $\hat{\delta}(q_0, w) = q$ if and only if $[q_0, w] \vdash^* [q, \lambda]$.
- So:

$$\mathcal{L}(M) = \{w \mid \hat{\delta}(q_0, w) \in F\}$$

Other application: specification of system behavior

Ancient coffee machine

- $\Sigma = \{10, 25, \mathit{coffee}\}$; $Q = \{q_0, \ldots, q_7\}$
- What is going on here with δ ?



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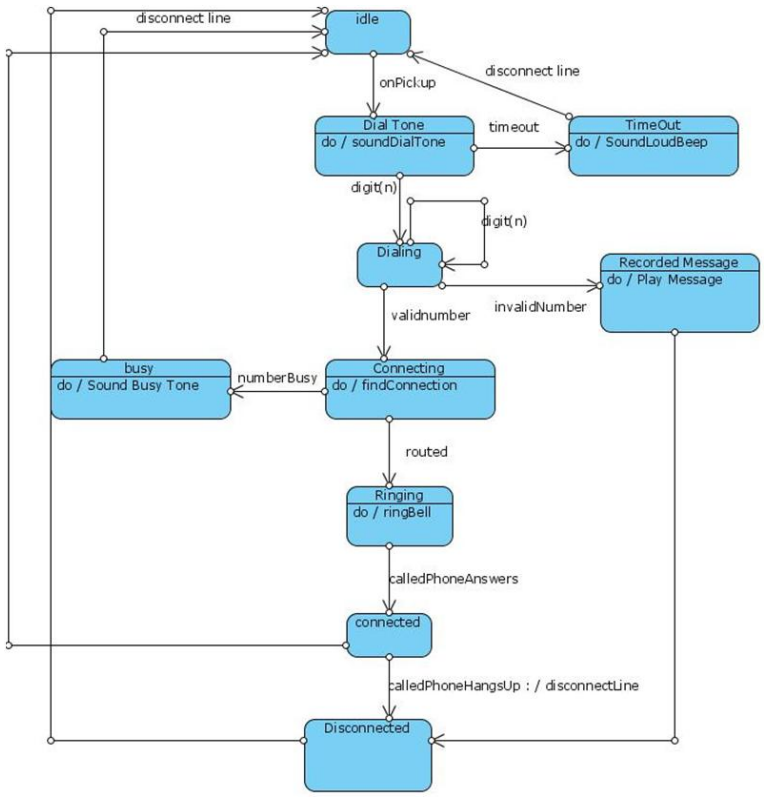
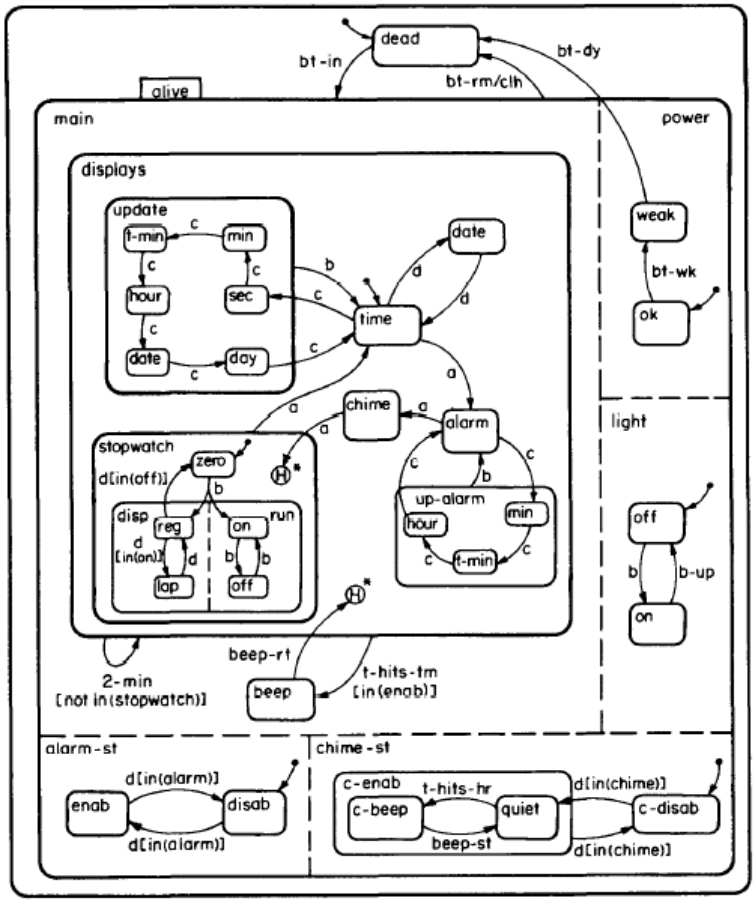


Telephone operation

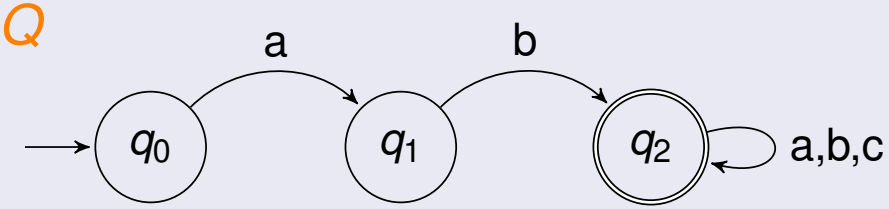


Digital Watch

Also: Communication Protocols, Railway Interlockings, Cell Biology

Incomplete deterministic finite automaton

- An incomplete automation has a partial function:
  $\delta : Q \times \Sigma \rightharpoonup Q$



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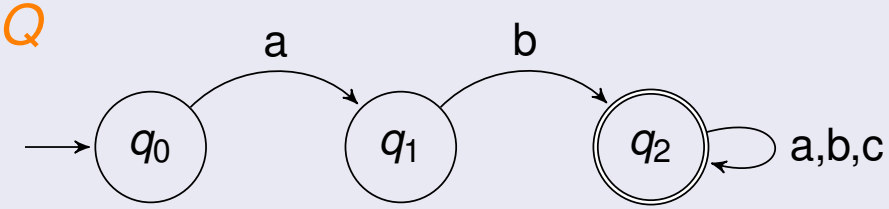

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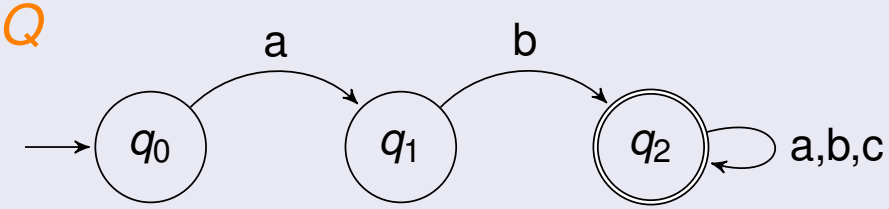

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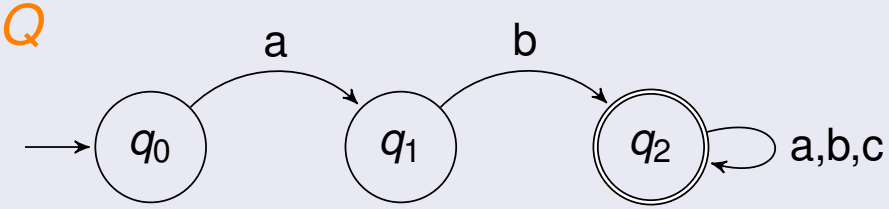

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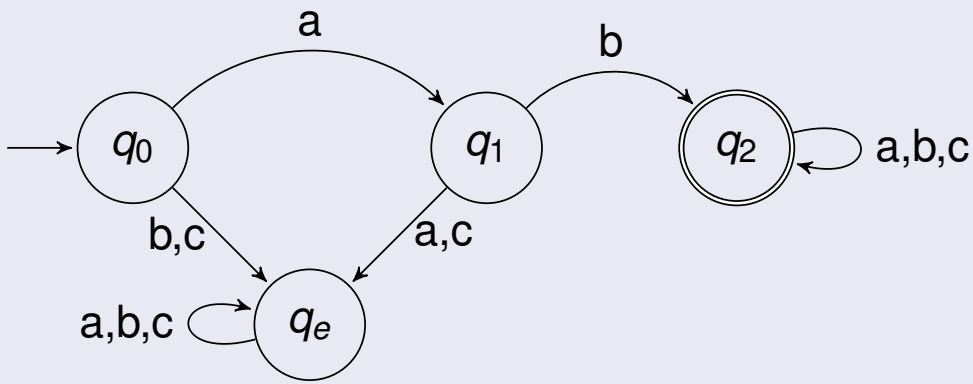

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Complete an incomplete DFA

Precise construction:

- Given an incomplete DFA

$$M = (Q, \Sigma, \delta, q_0, F)$$

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Observations

- M and M' are equivalent in the sense that $\mathcal{L}(M) = \mathcal{L}(M')$.
- An incomplete DFA requires less arrows, but the above shows that you cannot define other languages with it.

Complement DFA (construction)

Complementing a DFA $M = (Q, \Sigma, \delta, q_0, F)$

- Construct a corresponding DFA M' as follows:

$$M' = (Q, \Sigma, \delta, q_0, Q - F)$$

- Theorem (5.3.3): $\mathcal{L}(M') = \Sigma^* - \mathcal{L}(M) = \overline{\mathcal{L}(M)}$

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- If yes: then also closed under intersection: $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$

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Solution:

- Non-deterministic Finite Automata (NFA): Guess!

Discussed this lecture:

Languages:

- Alphabet, Words, Language, Operations

Regular Expressions:

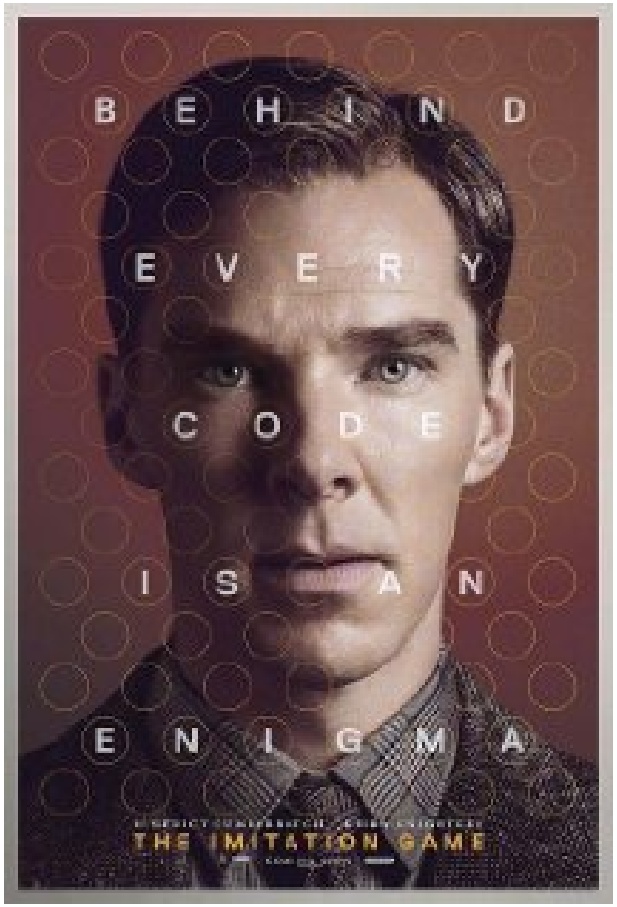
- Choice, Sequence, Kleene star
- Regular languages $\mathcal{L}(E)$

Deterministic Finite Automata (DFA):

- States, Transitions, Acceptance
- Accepted language $\mathcal{L}(M)$
- Complete versus incomplete (equally powerful)

Don’t miss out on this: The Imitation Game

Alan Turing



Benedict Cumberbatch