Algebra Sample Test

1. (a) (5 points) Compute the order of each element in $U(18)$.
   (b) (4 points) Prove that $U(18)$ is isomorphic to $U(14)$.

2. (a) (4 points) Prove that the ring R defined by

$$R = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$$

   is an integral domain.

   (b) (3 points) Is the ring S defined by

$$S = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$$

   an integral domain?

   (c) (2 points) Is the ring T defined by

$$T = \{a + b\sqrt{2} \mid a, b \in \mathbb{R}\}$$

   an integral domain?

3. We want to paint the edges of a square made of iron wire using red and blue. We want to use Burnside's theorem to determine the number of different colorings.

   (a) (3 points) What, in the terminology of Burnside's theorem, is the set S and what is the group of permutations G acting on S.
   (b) (4 points) Determine the number of orbits in S under G.
   (c) (4 points) Determine for each element in S the corresponding orbit.

4. Consider $p(x) \in \mathbb{Z}_3[x]$ defined by $p(x) = x^2 + 1$ and let $\mathbb{F}$ be defined as

$$\mathbb{F} = \mathbb{Z}_3[x]/ < p(x) > .$$

   (a) (3 points) Argue that $\mathbb{F}$ is a field.
   (b) (3 points) Describe the elements of $\mathbb{F}$.
   (c) (2 points) How many elements does $\mathbb{F}$ have.
   (d) (3 points) Prove that the multiplicative group $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$ is cyclic.