# Algebra 20-03-2019

$G$ is a ~~set~~ group of permutations of a set $S$

$\varphi \in G$: $\varphi : S \to S$ bijective

$H$ be a subgroup of $G$

$|H|$ divides $|G|$

$|G| / |H|$ is the number of cosets

$i \in S$

Take $H = \mathrm{Stab}_G(i) = \{\varphi \in G \mid \varphi(i) = i\}$

$H$ is a subgroup

$\mathrm{Orb}_G(i) = \{\varphi(i) \mid \varphi \in G\}$.

$$\boxed{\text{Thm } \ |\mathrm{Stab}_G(i)|\,|\mathrm{Orb}_G(i)| = |G|}$$

idea of proof Lagrange $|H|(\#\text{cosets}) = |G|$

$H = Stab_G(i)$, to show that #cosets $=$
$|Orb_G(i)|$

need bijection between $Orb_G(i)$ and cosets of $H$

a coset

$$\varphi\, Stab_G(i) \overset{T}{\longmapsto} \varphi(i) \in Orb_G(i)$$

if $\psi \in \varphi\, Stab_G(i)$ then $\psi = \varphi \cdot \varphi'$ $\varphi' \in Stab_G(i)$

$$\psi(i) = \varphi(\underbrace{\varphi'(i)}_{i}) = \varphi(i)$$

T is injective if $\varphi_1 Stab_G(i) \neq \varphi_2 Stab_G(i)$
then $\varphi_1(i) \neq \varphi_2(i)$.

T is also surjective

T is a bijective relation between set of cosets and the set of elements in the orbit

$i \in S$

$$Stab_G(i) = \{\varphi \in G \mid \varphi(i) = i\}$$

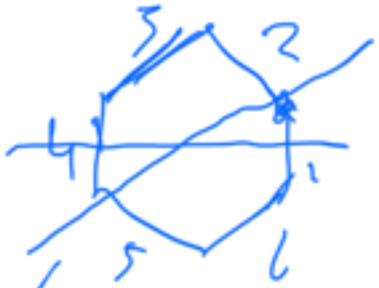
$Orb_G(i) = \{\varphi(i) \mid \varphi \in G\}$

$Orb_G(i) = \{\varphi(i) \mid \varphi \in G\}$.

$Fix_G(\varphi) = \{i \in S \mid \varphi(i) = i\}$.

Theorem (Burnside)

$$\text{number of orbits} = \frac{1}{|G|} \sum_{\varphi \in G} |fix_G(\varphi)|$$

Example 


$G = R_0, R_{60}, \ldots, R_{300}$

6 reflections

$G = D_6$

$|S| = 2^6 = 64$

$|fix(R_0)| = 64$

$|fix(R_{60})| = 2 = |fix(R_{300})|$

$|fix(R_{120})| = 2^2 = 4$

$= |fix(R_{240})| = 4$

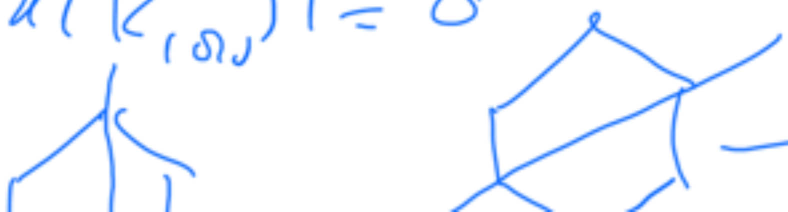
$|fix(R_{180})| = 8$

1 2 3 4 5 6
3 4 5 6 1

$(135)(246)$

4 5 6 1 2 3

$(14)(25)(36)$



3x

$|\mathrm{fix}(\varphi)| = 8 \cdot 3x$

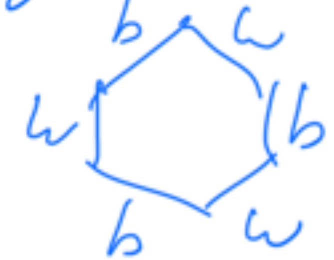
$|\mathrm{fix}(\varphi)| = 16$

$$(\underbrace{64 + 2\cdot 2 + 2\cdot 4 + 8 + 3\cdot 8 + 3\cdot 16}_{156}) =$$

64 4 8 8 24 48

total number of orbits $\dfrac{156}{12} = 13$

S is the set of all possible colourings disregarding symmetries


element of S