Algebra 13-3-2019:

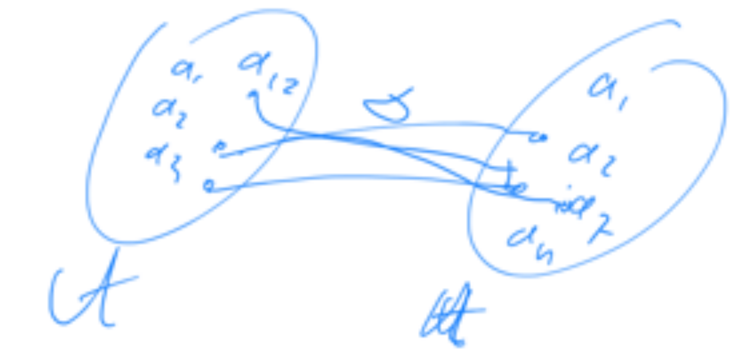
Permutations CH5

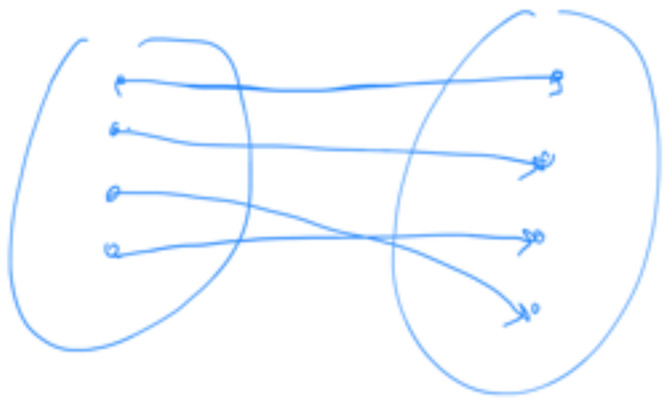
Let A be a finite set, a permutation of A is a map $\sigma: A \to A$:

σ injective

or σ surjective onto

$A = \{a_1, a_2, a_3, a_4, \dots a_n\}$.





injective $\Leftrightarrow$ surjective for finite sets

$\sigma: \mathbb{Z} \to \mathbb{Z}$ $\sigma(k) = 2k$

injective but not surjective

Example

$\alpha \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 4 & 2 \end{bmatrix}$

$\beta \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 5 & 4 \end{bmatrix}$

$\alpha \circ \beta \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 2 & 4 \end{bmatrix}$

$\beta \circ \alpha \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{bmatrix}$

All permutation of $\{1,2,3,4,5\} =: S_5$

$|S_5| = 120 = 5!$

$\varepsilon = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$

$\forall \alpha \in S_5: \quad \varepsilon \circ \alpha = \alpha \circ \varepsilon = \alpha$

$\forall \alpha \in S_5 \ \exists \beta \in S_5: \quad \alpha \circ \beta = \beta \circ \alpha = \varepsilon.$

$\forall \alpha, \beta, \gamma \in S_5: \quad (\alpha \circ \beta) \gamma = \alpha \circ (\beta \circ \gamma)$

$\Rightarrow (S_5, \cdot)$ is a group
non-commutative.

$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 4 & 2 \end{bmatrix}$

$\alpha = (1\ 3\ 5\ 2)(4) = (4)(1\ 3\ 5\ 2)$

$\beta = (1)(2\ 3)(4\ 5) = (2\ 3)(1)(4\ 5)$

$|\alpha| = 4$

$|\beta| = 2$

$\gamma = (1\ 3\ 4)(5\ 6)(2) \quad |\gamma| = 6$

$\delta = \underset{\uparrow}{\alpha} \cdot \underset{\uparrow}{\beta}$ $\quad |\alpha| = 4 \quad |\beta| = 6$

cyclic, disjoint $\quad \Rightarrow |\delta| = 12$

$|\delta| = \mathrm{lcm}(|\alpha|, |\beta|)$

Let $\alpha \in S_n$, then α can be written as:

$$\alpha = \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \dots \cdot \alpha_k$$

α_i cyclic and $\alpha_i \cap \alpha_j = \emptyset$ $i \neq j$

$\alpha_i \in S_n$.

proof by induction on the length of α.

$n = 8$ $\quad a_1 = 1 \quad a_2 = \alpha(a_1) \quad a_3 = \alpha(a_2)$

$a_k = \alpha(a_{k-1})$

$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 8 & 5 & 4 & 1 & 6 & 7 \end{matrix}$

(i) $\{a_1, \dots, a_k\} = \{1, \dots, n\}$.

$\Rightarrow \alpha$ cyclic

$a_1 = 1, a_2 = 3 \quad a_3 = 8$

$a_4 = 7, a_5 = 6, a_6 = 1 = a_1$

(ii) $\{a_1, \dots, a_k\} \subsetneq \{1, \dots, n\}$.

$\underbrace{(1\ 3\ 8\ 7\ 6)}(2)(4\ 5)$

$\Rightarrow \exists k$ s.t. $a_{k+1} \in \{a_1, \dots, a_k\}$.

$a_{k+1} = \alpha(a_k)$ $\alpha(a_k) = a_1$

$\alpha = (a_1 a_2 a_3 \dots a_k) \cdot \underset{\uparrow}{\beta}$

induction $\Rightarrow \beta = \beta_1 \dots \beta_l$

cyclic disjoint

$\alpha = (a_1 \dots a_{k-1}) \beta_1 \dots \beta_l$.

$\square$

$(1\ 3\ 8\ 7\ 6) = (1\ 6)(1\ 7)(1\ 8)(1\ 3)$

$(1\ 6)\underbrace{(8\ 7)(8\ 7)}_{\varepsilon}(1\ 7)(1\ 8)(1\ 3)$

cyclic $(a_1\ a_2\ a_3 \dots a_{n-1}\ a_n)$

$= (a_1\, a_n)(a_1\, a_{n-1})(a_1\, a_{n-2}) \cdots (a_1\, a_3)(a_1\, a_2)$

$\alpha \in S_n \quad \alpha = \alpha_1 \cdots \alpha_k \quad \alpha_i$ cyclic

$\alpha_1 = \beta_{11}\beta_{12}\beta_{13} \cdots \beta_1 \ldots$

$\alpha_2 = \beta_{21}\beta_{22}\beta_{23} \cdots \beta_2 \ldots$

Theorem. Let $\alpha = \alpha_1\alpha_2 \cdots \alpha_s = \beta_1\beta_2 \cdots \beta_t$

α_i, β_j 2-cycles

then $s+t$ is even

Lemma $\varepsilon = \beta_1 \cdots \beta_n \quad \beta_i$ 2-cycle

n even.

proof Thm. $\alpha_1\alpha_2 \cdots \alpha_s = \beta_1\beta_2 \cdots \beta_t$

$\Rightarrow \alpha_1\alpha_2 \cdots \alpha_s \beta_t \beta_{t-1}\beta_{t-2} \cdots \beta_1 = \varepsilon$

lemma $\Rightarrow$ $s+t$ is even.

$\varepsilon = \beta_1\beta_2 \cdots \beta_{n-1}\beta_n$

$\beta_n = (a\,b) \qquad \beta_{n-1} = (a\,b) \leftarrow \beta_{n-1}\beta_n = \varepsilon$

$\beta_{n-1}\beta_n = (a\,c)(a\,b) \qquad = (a\,c) \qquad \beta_{n-1}\beta_n = (ab)(bc)$

$a\,b\,c \qquad = (b\,d)$

$b\,c\,a \qquad (c\,d) \qquad = (ab)(cd)$

$(a\,b\,c) = (b\,c\,a) = (b\,a)(b\,c)$

So we can define α even if s even

α odd if s odd

Let α, β even permutations:

then $\alpha \cdot \beta$ also even

ε is even

$\Rightarrow$ subgroup of S_n: A_n (alternating group)

$|S_n| = n! \qquad |A_n| = \frac{n!}{2}$

If α is even then $112/\alpha$ is odd