Burnside’s theorem

Let G be a group of permutations of a finite set S: $\varphi \in G \Rightarrow \varphi : S \to S$, bijective.

Define:

$s \in S$: $Stab_G(s) = \{\varphi \in G \mid \varphi(s) = s\}$.

$Orb_G(s) = \{\varphi(s) \mid \varphi \in G\}$

$Fix_G(\varphi) = \{s \in S \mid \varphi(s) = s\}$.

Denote the different orbits by $S_1, \dots, S_M$.

Notice that $S_i \cap S_j = \emptyset$ $i \neq j$ and $S = \bigcup_{j=1}^{M} S_j$, hence the orbits form a partition of S.

Notice furthermore that $t \in Orb_G(s)$ implies $Orb_G(s) = Orb_G(t)$.

Theorem (Burnside)

$$M = \frac{1}{|G|} \sum_{\varphi \in G} |\mathrm{fix}_G(\varphi)|$$

number of orbits

Proof Define $\Omega = \{(\varphi, s) \in G \times S \mid \varphi(s) = s\}$.

$$\Omega = \bigcup_{\varphi \in G} \{s \in S \mid \varphi(s) = s\} = \bigcup_{\varphi \in G} \mathrm{fix}_G(\varphi) \qquad (1)$$

but also:

$$\Omega = \bigcup_{s \in S} \{\varphi \in G \mid \varphi(s) = s\} = \bigcup_{s \in S} \mathrm{Stab}_G(s) \qquad (2)$$

From (1) it follows that

$$|\Omega| = \sum_{\varphi \in G} |\mathrm{fix}_G(\varphi)| \qquad (3)$$

From (2): $$|\Omega| = \sum_{s \in S} |\mathrm{Stab}_g(s)| \qquad (4)$$

hence $$|\Omega| = \sum_{s \in S} |\mathrm{Stab}_g(s)| = \sum_{k=1}^{M} \sum_{s \in S_k} |\mathrm{Stab}_g(s)|$$

$$= \sum_{k=1}^{M} \sum_{s \in S_k} \frac{|g|}{|\mathrm{Orb}_g(s)|} = |g| \cdot \sum_{k=1}^{M} \sum_{s \in S_k} \frac{1}{|\mathrm{Orb}_g(s)|}$$

$$= |g| \sum_{k=1}^{M} \sum_{s \in S_k} \frac{1}{|S_k|} = \sum_{k=1}^{M} \frac{|S_k|}{|S_k|} = M .$$

This proves the statement

$\square$