Solutions to ADC practice questions: induction

Question 1.

Prove that $\sum_{i=1}^{n} i^2 = \frac{2n^3 + 3n^2 + n}{6}$ for $n \geq 1$.

Solution

The proof is by induction on n.

Base case. The base case is $n = 1$. The following then holds

$$\sum_{i=1}^{n} i^2 = \sum_{i=1}^{1} i^2 = 1^2 = 1$$

Moreover, the following holds

$$\frac{2n^3 + 3n^2 + n}{6} = \frac{2 + 3 + 1}{6} = 1$$

So equality holds in the base case.

Inductive step. Let n be any number larger than or equal to 1 and assume that the equality holds for all $1 \leq k \leq n$. So the induction hypothesis is that the following formula holds for all $1 \leq k \leq n$.

$$\sum_{i=1}^{k} i^2 = \frac{2k^3 + 3k^2 + k}{6}$$

From this, the following can be derived:

$$\begin{aligned}
\sum_{i=1}^{n+1} i^2 &= \sum_{i=1}^{n} i^2 + (n+1)^2 \\
&= \frac{2n^3 + 3n^2 + n}{6} + (n+1)^2 \\
&= \frac{2n^3 + 3n^2 + n}{6} + n^2 + 2n + 1 \\
&= \frac{2n^3 + 3n^2 + n}{6} + \frac{6n^2 + 12n + 6}{6} \\
&= \frac{2n^3 + 9n^2 + 13n + 6}{6}
\end{aligned}$$

Moreover, the following holds

$$\begin{aligned}
\frac{2(n+1)^3 + 3(n+1)^2 + n + 1}{6} &= \frac{2(n^3 + 3n^2 + 3n + 1) + 3(n^2 + 2n + 1) + n + 1}{6} \\
&= \frac{2n^3 + 9n^2 + 13n + 6}{6}
\end{aligned}$$

So equality also holds for $n + 1$.

Question 2.

The Fibonacci sequence is defined as

$$F(n) = F(n-1) + F(n-2) \qquad \text{voor } n \geq 2$$

Starting with $F(0) = 0$ en $F(1) = 1$.

Prove, by induction, that: $F(n) \geq 0.01 \left(\frac{3}{2}\right)^n$ for all $n \geq 1$.

Solution

The proof is by induction on n.

Base case. There are two base cases, $n = 1$ and $n = 2$. The following holds:

$$F(1) = 1 \geq 0.015 = 0.01 \left(\frac{3}{2}\right)^1$$

$$F(2) = 0 + 1 = 1 \geq 0.0225 = 0.01 \left(\frac{3}{2}\right)^2$$

So the inequality holds for the base cases.

Inductive step. Let n be any number larger than 2, and assume that the equality holds for all $1 \leq k < n$. So the induction hypothesis is that the following formula holds for all $1 \leq k < n$

$$F(k) \geq 0.01 \left(\frac{3}{2}\right)^k$$

From this, the following can be derived:

$$\begin{aligned} F(n) &= F(n-1) + F(n-2) \\ &\geq 0.01 \left(\frac{3}{2}\right)^{n-1} + 0.01 \left(\frac{3}{2}\right)^{n-2} \\ &= 0.01 \left(\frac{3}{2}\right)^n \cdot \frac{2}{3} + 0.01 \left(\frac{3}{2}\right)^n \cdot \frac{4}{9} \\ &= 0.01 \left(\frac{3}{2}\right)^n \left(\frac{2}{3} + \frac{4}{9}\right) \\ &> 0.01 \left(\frac{3}{2}\right)^n \end{aligned}$$

So the inequality also holds for all $n > 2$.