Artificial Intelligence
Answers to exercises for Tutorial 2 on Predicate Logic

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Introduction

FOPL stands for First Order Predicate Logic.

Exercises

1. Aristotle studied reasoning with noun phrases like "every man", "some man" and "no man". The four sentence forms Aristotle treated in his syllogisms theory are:

   (a) All P's are Q's (e.g., all men are mortal)

   (b) Some P's are Q's (e.g., some men are mortal)

   (c) No P's are Q's (e.g., no men is mortal)

   (d) Some P's are not Q's (e.g., some men are not mortal)

   Translate these four sentences into well-formed formulas of FOPL.

   Answer:

   a All P's are Q's (e.g., all men are mortal) $\forall x(P(x) \Rightarrow Q(x))$ (and not: $\forall x(P(x) \wedge Q(x))$, which says, e.g., that all individuals are men and they are mortal).

   b Some P's are Q's (e.g., some men are mortal) $\exists x(P(x) \wedge Q(x))$ (and not: $\exists x(P(x) \Rightarrow Q(x))$ which says, e.g., that there is an individual for which it holds that if they are a man, they are mortal; this sentence would also be true if there is an individual who is not a man, since for this individual the implication holds due to the antecedent being false).

   c No P's are Q's (e.g., no men is mortal) $\forall x(P(x) \Rightarrow \neg Q(x))$ (or, equivalently, $\neg\exists x(P(x) \wedge Q(x))$, which we can show to be equivalent as follows: $\neg\exists x(P(x) \wedge Q(x)) \equiv \forall x \neg(P(x) \wedge Q(x)) \equiv \forall x(\neg P(x) \vee \neg Q(x)) \equiv \forall x(P(x) \Rightarrow \neg Q(x)))$

d Some P's are not Q's (e.g., some men are not mortal) $\exists x(P(x) \wedge \neg Q(x))$.

2. Translate the following sentences into FOPL:

   (a) "A small, happy dog is at home"

   (b) "Every small dog that is at home is happy"

   Answer: (a) $\exists x(Small(x) \wedge Happy(x) \wedge Dog(x) \wedge Home(x))$.

   (b) $\forall x[(Small(x) \wedge Dog(x) \wedge Home(x)) \Rightarrow Happy(x)]$.

3. Given are the following predicates:

   - $P(x)$
   - $Q(x)$

   Now consider the following sentence: $\neg\forall x\ (P(x) \wedge Q(x))$. Which of the following models satisfies this sentence?

   (a) No objects and relations P, Q.

   (b) One object a and relations P, Q.

   (c) One object a and relations P, Q with $a \in P$ and $a \in Q$.

   Answer (b). $\neg\forall x\ (P(x) \wedge Q(x)) \equiv \exists x\ \neg(P(x) \wedge Q(x)) \equiv \exists x\ (\neg P(x) \vee \neg Q(x))$, i.e., there must be at least one object for which it holds that it is not in the relation P or not in the relation Q. (a) has no objects, while (c) has one object which is in both relations. (b) has one object which is in neither relation, so the model satisfies the sentence.

4. Given are the following three predicates:

   - $Food(x)$: x is something to eat.
   - $Person(x)$: x is a person.
   - $Likes(x, y)$: x likes y

   Now consider the following two sentences in first-order logic:

   (I) $\forall x\ (Food(x) \Rightarrow \exists y\ (Person(y) \wedge Likes(y, x)))$
   (II) $\exists x\ (Food(x) \wedge \forall y\ (Person(y) \Rightarrow Likes(y, x)))$

   Which of the following statements about these two sentences is correct? Construct simple models, i.e., a set of objects and a relation for each predicate (i.e., a relation symbol and a specification of which objects are part of that relation), over which the truth of the sentences can be evaluated, to find an answer.

(a) (I) and (II) are equivalent.

(b) (I) and (II) are not equivalent, but (I) follows from (II).

(c) (I) and (II) are not equivalent, but (II) follows from (I).

(d) (I) and (II) are not equivalent, and neither follows from the other.

Correct answer (d).

If one sentence ψ follows from another sentence ϕ, just like in propositional logic it must be the case that for all models M that satisfy ϕ (i.e., $M \models \phi$), it must be the case that $M \models \psi$. Thus to show that ψ does not follow from ϕ, it suffices to find *one* model for which it holds that $M \models \phi$ but $M \not\models \psi$. If sentences are equivalent, they have the same models.

A model for a predicate logic formula consists of a set of objects and a relation for each predicate. Let's start with a model M with two objects, *John* and *Bread* and relations *Food*, *Person* and *Likes* with $Bread \in Food$, $John \in Person$ and $(John, Bread) \in Likes$. For this model we have $M \models I$. This model however also satisfies II. However, if we introduce another Person $Mary \in Person$ the second sentence does not hold since Mary does not like bread and bread is the only food, i.e., we must have $\{x/Bread\}$. Thus we have that $M \not\models II$.

Vice versa, we construct a model M' for which II holds. Let M' have two objects, *John* and *Bread* and relations *Food*, *Person* and *Likes* with $Bread \in Food, John \in Person$ and $(John, Bread) \in Likes$. For this model we have $M' \models II$. However, this model also satisfies I. If now introduce another food, $Pizza \in Food$, the first sentence does not hold since for pizza there is no one who likes it. Therefore $M \not\models I$.

5. **Not for Create**
Consider the two statement in the language of first-order logic:

I.) $\forall x(P(x) \Rightarrow \exists y Q(y))$
II.) $\forall x \neg P(x) \vee \neg \forall y \neg Q(y)$

The question is whether there is an entailment relation between these two sentences. We write $P \models R$ for "P entails R" and $P \not\models R$ for "P does not entail R".Only one of the following is true; which one?

(a) I $\models$ II but II $\not\models$ I.

(b) I $\not\models$ II but II $\models$ I.

(c) I $\not\models$ II and II $\not\models$ I.

(d) I $\models$ II and II $\models$ I.

Remember that, for any two sentences in first order logic φ and ψ, $\varphi \vDash \psi$ holds if $\varphi \vdash \psi$. You can check whether $\varphi \vdash \psi$ by means of resolution: find out whether

$\varphi \wedge \neg\psi$ is a contradiction. For checking whether I $\models$ II and II $\models$ I you need to apply the resolution procedure two times.

Alternatively, you may try to prove directly by rewriting the formulas using logical equivalences.

Answer: d. $\forall x(P(x) \Rightarrow \exists y Q(y)) \equiv \forall x(\neg P(x) \vee \exists y Q(y)) \equiv \forall x(\neg P(x) \vee \neg\forall y \neg Q(y)) \equiv \forall x \neg P(x) \vee \neg\forall y \neg Q(y)$. The latter step is allowed because $(\forall x P) \vee Q \Leftrightarrow \forall x[P \vee Q]$ whenever x not free in Q, i.e., whenever x does not occur in Q without being quantified over. If the latter would be the case, it would be brought within the scope of the outermost $\forall$ quantifier, which changes the meaning of the formula.

6. **Not for Create**
Consider the formula $Knows(John, x)$. We try to unify this formula with different other formula. In one of these cases unification fails. For which one unification will fail?

   (a) $Knows(John, Jane)$
   (b) $Knows(y, Bill)$
   (c) $Knows(y, Mother(y))$
   (d) $Knows(x, Jane)$

   Answer: d, since x would have to be unified with both John and Jane, which is not possible. When using resolution, name clash should be solved by standardizing apart, i.e., renaming variables such that they are different per sentence.

7. **Not for Create**
We want to Skolemise the following sentence in first-order logic:

$$\forall x \forall y \exists z \forall v \ (P(x, y) \wedge Q(v) \Rightarrow R(x, y, z, v))$$

   Of the following four substitutions, only one produces a correct Skolemisation. Which one?

   (a) $\{z/S\}$
   (b) $\{z/S(x)\}$
   (c) $\{z/S(v)\}$
   (d) $\{z/S(x, y)\}$

   The idea is that you correctly choose whether to supply a Skolem constant or a Skolem function. If the answer is a Skolem function, you further have to choose of which variables it is a function.

   answer: d. The existential quantifier $\exists z$ occurs within the scope of two universal quantifiers, namely for the variables x and y. Since z occurs in a relation where also x and y are parameters, z should be replaced by a Skolem function parameterized with x and y.

8. **Not for Create**
Let $Child(x, y, z)$ mean x is child of y and z.
(a) Express in FOPL that a child of z and y is a child of y and z.
(b) Let a knowledge base KB contain this formula as well as the fact $Child(Eve, Anne, Oscar)$. Prove using natural deduction (=applying inference rules) that $Child(Eve, Oscar, Anne)$. Apply the rules of elimination of quantifiers (instantiation) and Modus Ponens.

Answer: (a) $\forall x, y, z : (Child(x, y, z) \Rightarrow Child(x, z, y))$.
(b):

1. $\forall x, y, z : (Child(x, y, z) \Rightarrow Child(x, z, y))$ (KB)
2. $Child(Eve, Anne, Oscar)$ (KB)
3. $Child(Eve, Anne, Oscar) \Rightarrow Child(Eve, Oscar, Anne)$ (1, elimination of universal quantifiers)
4. $Child(Eve, Oscar, Anne)$ (2,3, Modus Ponens)

9. **Not for Create**
A knowledge base (KB) contains the two statements:

$$\forall x(P(x, b) \vee Q(x)) \qquad (1)$$

$$\forall y(\neg P(f(y), b) \vee Q(y)) \qquad (2)$$

Show (taking the steps of a proof by resolution) that the following is derivable from this KB.

$$\forall y(Q(y) \vee Q(f(y))) \qquad (S)$$

Answer:

Step 1: Negate S: into prenex normal form:

$$\exists y[\neg Q(y) \wedge \neg Q(f(y))]$$

Step 2: Skolemize:

$$\neg Q(c) \wedge \neg Q(f(c))$$

These are the two sentences we add to the KB, so that we have 4 clauses:

(1) $P(x, b) \vee Q(x)$,
(2) $\neg P(f(y), b) \vee Q(y)$,

(3) $\neg Q(c)$ and

(4) $\neg Q(f(c))$.

Step 3: Resolution.

(5). From 1,2 - substitute $f(y)$ for x - resolvent $Q(y) \vee Q(f(y))$

(6). From 3,5 - substitute c for y - resolvent $Q(f(c))$

(7). From 4,6 derive empty clause (contradiction).