Discrete Mathematics for Computer Science, Part 2; Sample Test 2. Solutions/Correction standard

1. Basis step for $n = 60$:
   $60 = 10 \cdot 6$, so take $s = 10$ and $t = 0$. Thus the statement is correct for $n = 60$. **[1 pt]**
   Induction step:
   Let $k \geq 60$ and suppose that:
   $k = 6s + 13t$ for some $s, t \in \mathbb{N}$ (Induction Hypothesis: IH). **[1 pt]**

   We must show that IH implies: $k + 1 = 6a + 13b$ for some $a, b \in \mathbb{N}$. **[1 pt]**

   Well, by IH, we have: $k + 1 = 6s + 13t + 1$ for some $s, t \in \mathbb{N}$. **[1 pt]**
   Now we distinguish the cases $s \geq 2$ and $s < 2$.
   If $s \geq 2$, then $k + 1 = 6(s - 2) + 6 \cdot 2 + 13t + 1 = 6(s - 2) + 13(t + 1)$, so take $a = s - 2$ and $b = t + 1$. **[1 pt]**
   If $s < 2$, then $t \geq 5$ because $k \geq 60$.
   So then $k + 1 = 6s + 13(t - 5) + 13 \cdot 5 + 1 = 6(s + 11) + 13(t - 5)$, so take $a = s + 11$ and $b = t - 5$. **[1 pt]**
   So in any case we showed that $k + 1 = 6a + 13b$ for some $a, b \in \mathbb{N}$, which completes the proof by the principle of mathematical induction.
   (From the proof it must be crystal clear whàt is supposed (1 pt) and whàt must be proved (1 pt). In case the induction hypothesis is not correctly formulated or the proof is not clearly written down: at most 1 pt for the entire exercise)

2. (i) Proof of $f^{-1}(B_1 \cap B_2) \subseteq f^{-1}(B_1) \cap f^{-1}(B_2)$:
   Let $x \in f^{-1}(B_1 \cap B_2)$. Then $f(x) \in B_1 \cap B_2$.
   So $f(x) \in B_1$ and $f(x) \in B_2$.
   Therefore: $x \in f^{-1}(B_1)$ and $x \in f^{-1}(B_2)$.
   And so: $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$. **[3 pt]**

   (ii) Proof of $f^{-1}(B_1) \cap f^{-1}(B_2) \subseteq f^{-1}(B_1 \cap B_2)$:
   Let $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$. Then $x \in f^{-1}(B_1)$ and $x \in f^{-1}(B_2)$.
   So $f(x) \in B_1$ and $f(x) \in B_2$, and therefore $f(x) \in B_1 \cap B_2$.
   Hence: $x \in f^{-1}(B_1 \cap B_2)$. **[3 pt]**

3. (i) R is reflexive, since for all $(x_1, y_1) \in A$ we have:
   $(x_1, y_1)R(x_1, y_1)$, since $x_1 y_1 = y_1 x_1$. **[1 pt]**

   (ii) R is symmetric, since for all $(x_1, y_1), (x_2, y_2) \in A$ we have:
   if $(x_1, y_1)R(x_2, y_2)$ then $x_1 y_2 = x_2 y_1$, so $x_2 y_1 = x_1 y_2$ and hence $(x_2, y_2)R(x_1, y_1)$. **[1 pt]**

   (iii) R is transitive, since for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in A$ we have:
   if $(x_1, y_1)R(x_2, y_2)$ and $(x_2, y_2)R(x_3, y_3)$ then $x_1 y_2 = x_2 y_1$ and $x_2 y_3 = x_3 y_2$.
   Multiplying the first equation with y_3 yields: $x_1 y_2 y_3 = x_2 y_1 y_3$ and so, by the second equation: $x_1 y_2 y_3 = x_3 y_1 y_2$.
   Dividing by $y_2 \neq 0$ yields: $x_1 y_3 = x_3 y_1$ and hence $(x_1, y_1)R(x_3, y_3)$. **[2 pt]**

   (iv) The partition of A induced by R consists of the equivalence-classes of R.
   We have for $(x_1, y_1) \in A$:
   $[\,(x_1, y_1)\,] = \{(x_2, y_2) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid x_1 y_2 = x_2 y_1\} = \{(x_2, y_2) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid y_2 = \frac{y_1}{x_1} x_2\}$.
   So $[\,(x_1, y_1)\,]$ is the line through $(0, 0)$ in the first quadrant, with slope $\frac{y_1}{x_1}$.
   Hence, the partition P of A induced by R consists of all lines through $(0, 0)$ in the first quadrant. **[2 pt]**