

Discrete Mathematics for Computer Science,
Part 2; Sample Test 1. Solutions/Correction standard

1. Proof with Mathematical Induction (alternative form; Theorem 4.2).

Basis step for $n = 1$, $n = 2$ and $n = 3$:

$$a_1 = 2 \leq \left(\frac{5}{2}\right)^1; a_2 = 6 \leq \left(\frac{25}{4}\right) = \left(\frac{5}{2}\right)^2; a_3 = 15 \leq \left(\frac{125}{8}\right) = \left(\frac{5}{2}\right)^3.$$

Thus the statement is correct for $n = 1$, $n = 2$ and $n = 3$.

[1 pt]

Induction step:

Let $k \geq 3$ and suppose that for all $1 \leq i \leq k$:

$$a_i \leq \left(\frac{5}{2}\right)^i \quad (\text{Induction Hypothesis: IH}).$$

[1 pt]

We must show that IH implies:

$$a_{k+1} \leq \left(\frac{5}{2}\right)^{k+1}.$$

[1 pt]

Well, we have

$$a_{k+1} = a_k + 2a_{k-1} + 4a_{k-2} \leq \left(\frac{5}{2}\right)^k + 2\left(\frac{5}{2}\right)^{k-1} + 4\left(\frac{5}{2}\right)^{k-2},$$

where the inequality follows from the Induction Hypothesis.

[1 pt]

Now it suffices to show that

$$\left(\frac{5}{2}\right)^k + 2\left(\frac{5}{2}\right)^{k-1} + 4\left(\frac{5}{2}\right)^{k-2} \leq \left(\frac{5}{2}\right)^{k+1}.$$

By dividing both sides of this inequality by $\left(\frac{5}{2}\right)^{k-2}$, we obtain the equivalent inequality

$$\left(\frac{5}{2}\right)^2 + 2\left(\frac{5}{2}\right)^1 + 4 \leq \left(\frac{5}{2}\right)^3.$$

This inequality clearly holds, since

$$\frac{25}{4} + \frac{10}{2} + 4 = \frac{50 + 40 + 32}{8} = \frac{122}{8} \leq \frac{125}{8}.$$

[2 pt]

This completes the proof by the principle of mathematical induction.

(From the proof it must be crystal clear what is supposed [1 pt] and what must be proved [1 pt].

In case the induction hypothesis is not correctly formulated or the proof is not clearly written down: at most **1 pt** for the entire exercise)

2. (i) f is commutative, since for all $A, B \in \mathcal{P}(U)$ we have

$$f(A, B) = \overline{A \cup B} = \overline{B \cup A} = f(B, A).$$

[2 pt]

- (ii) f is *not* associative. E.g. take $U = \{1\}$; $A = \{1\}$; $B = C = \emptyset$, then

$$f(f(A, B), C) = \overline{\overline{A \cup B} \cup C} = \{1\} \quad \text{but} \quad f(A, f(B, C)) = \overline{A \cup \overline{B \cup C}} = \emptyset. \quad [2 \text{ pt}]$$

(iii) Suppose D is an identity of f . Then

$$f(D, A) = f(A, D) = A \quad \text{for all } A \in \mathcal{P}(U).$$

So $\overline{D \cup A} = A$ for all $A \in \mathcal{P}(U)$. Taking $A = \emptyset$ yields $\overline{D} = \emptyset$. Therefore $D = U$ is the only candidate for an identity. But $D = U$ yields $\overline{U \cup A} = A$ for all $A \in \mathcal{P}(U)$. Taking $A = U$, we get $U = \emptyset$, contradicting the assumption that U is nonempty. Hence f does not have an identity. [2 pt]

3. Let $n = |A|$ and write $A = \{a_1, a_2, \dots, a_n\}$.

(i) Proof of " \Rightarrow ":

Suppose that R is transitive.

Recall that $M_{ij} \in \{0, 1\}$ for all i, j and that Boolean addition ($1 + 1 = 1$) is used to compute M^2 . Hence, in order to prove that $M^2 \leq M$, it suffices to show that:

$$\text{if } M_{ij}^2 = 1 \text{ for some } i, j \in \{1, 2, \dots, n\}, \text{ then also } M_{ij} = 1.$$

So suppose $M_{ij}^2 = 1$ for some $i, j \in \{1, \dots, n\}$.

Then the dot product of the i^{th} row and j^{th} column of M is equal to 1: $\sum_{k=1}^n M_{ik}M_{kj} = 1$.

Therefore $\exists_{k \in \{1, 2, \dots, n\}} [M_{ik}M_{kj} = 1]$, so $\exists_{k \in \{1, 2, \dots, n\}} [M_{ik} = 1 \wedge M_{kj} = 1]$.

Hence, $\exists_{k \in \{1, 2, \dots, n\}} [a_i R a_k \wedge a_k R a_j]$, and therefore $a_i R a_j$, because R is transitive.

And so: $M_{ij} = 1$. [3 pt]

(ii) Proof of " \Leftarrow ":

Suppose that $M^2 \leq M$.

In order to prove that R is transitive, let $a_i, a_k, a_j \in A$ be such that $a_i R a_k$ and $a_k R a_j$.

We will show that $a_i R a_j$.

Well: $a_i R a_k$ and $a_k R a_j$ imply that $M_{ik} = 1$ and $M_{kj} = 1$, and so the dot product of the i^{th} row and j^{th} column of M is equal to 1 (note: here Boolean addition is used).

So $M_{ij}^2 = 1$, and therefore $M_{ij} = 1$ because $M^2 \leq M$.

Hence: $a_i R a_j$. [3 pt]