# Discrete Mathematics for Computer Science

## Lecture 13

# §7.4
# Equivalence relations and partitions

**Definition:** Let *A* be a set.

A set $P = \{A_1, A_2, \cdots, A_n\}$

is a **partition** of *A* if it satisfies the following three conditions:

(i) $\varnothing \neq A_i \subseteq A$ for all $i \in \{1, 2, \ldots, n\}$

(ii) $\bigcup_{i=1}^{n} A_i = A$

(iii) $A_i \cap A_j = \varnothing$ if $i \neq j$

The sets $A_i$ are the **blocks** of the partition

# Partitions $\{A_1, A_2, \cdots, A_n\}$

(i) $\varnothing \neq A_i \subseteq A$ for all $i \in \{1, 2, \ldots, n\}$

(ii) $\bigcup_{i=1}^{n} A_i = A$ (iii) $A_i \cap A_j = \varnothing$ if $i \neq j$

**Example:** $A = \{1,2,3,4,5,6,7\}$

Partitions of *A* are (among others):

$\{\{1,2\}, \{3\}, \{4,5,6\}, \{7\}\}$ $\{\{1,3,7\}, \{2,5\}, \{4,6\}\}$ $\{\{1,2,3,4,5,6,7\}\}$

$\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}\}$

The following sets are not partitions of *A*:

$\{\{1,3,7\}, \varnothing, \{2,5\}, \{4,6\}\}$ $\{\{1,3,7\}, \{2,5,8\}, \{4,6\}\}$

$\{\{1,7\}, \{2,5\}, \{4,6\}\}$ $\{\{1,3,7\}, \{2,5\}, \{4,5,6\}\}$

# Partitions

**Definition (generalization):**

**Let *A* be a set and let *I* an index-set.**

**A collection of sets** $\{A_i\}_{i\in I}$ **is a partition of *A* if it satisfies the following three conditions:**

**(i)** $\varnothing \neq A_i \subseteq A$ **for all** $i\in I$

**(ii)** $\bigcup_{i\in I} A_i = A$ **(iii)** $A_i \cap A_j = \varnothing$ **if** $i \neq j$

# Partition $\{A_i\}_{i\in I}$

(i) $\varnothing \neq A_i \subseteq A$ or all $i\in I$

(ii) $\bigcup_{i\in I} A_i = A$ (iii) $A_i \cap A_j = \varnothing$ if $i \neq j$

Examples: $A = Z$ $I = \{2k \mid k \in Z\}$

Then $\{\{i, i+1\} \mid i \in I\}$ is a partition of $A$

and $\{\{-i, i\} \mid i \in I\}$ is not a partition of $A$

# Equivalence Relations

**Definition:**

A relation $R$ on $A$ is called an **equivalence relation** if:

$R$ is reflexive, symmetric and transitive.

**Examples:**

The modulo 11 -relation on $Z$ is an equivalence relation.

The “ $\leq$ -relation on $Z$ ” is not an equivalence relation (since it is not symmetric).

# Equivalence Classes

**Definition:**

**Let $R$ be an equivalence relation on a set $A$.**

**For each $x \in A$ , the equivalence class $[x]$ is given by:**

$$[x] = \{ y \in A \mid (y, x) \in R \}$$

**Example:** **The modulo 11 -equivalence relation on $Z$.**

**Then** $[3] = \{\cdots, -30, -19, -8, 3, 14, 25, \cdots\}$ $= \{11k + 3 \mid k \in Z\}$

**also** $[-19] = \{\cdots, -30, -19, -8, 3, 14, 25, \cdots\}$ $= \{11k + 3 \mid k \in Z\}$

**and** $[1] = \{\cdots, -32, -21, -10, 1, 12, 23, \cdots\}$ $= \{11k + 1 \mid k \in Z\}$

# Theorem

$$[x] = \{y \in A \mid (y, x) \in R\}$$

**Let *R* be an equivalence relation on a set *A*.**
**Let *x*, *y*∈*A*. Then**

**(a)** $x \in [x]$

**(b)** $(x, y) \in R \quad \Leftrightarrow \quad [x] = [y]$

**(c)** $[x] = [y] \quad \text{or} \quad [x] \cap [y] = \varnothing$

**Proof: (a)**

**Since *R* is reflexive (equivalence relation), we have (*x*, *x*) ∈ *R*. So *x*∈[*x*] (by definition of [*x*])**

(a) $x \in [x]$ (b) $(x, y) \in R \quad \Leftrightarrow \quad [x] = [y]$

# Proof of (b)

"$\Rightarrow$": Suppose $(x, y) \in R$

If $w \in [x]$, then $(w, x) \in R$.

Together with $(x, y) \in R$ it follows from the transitivity of $R$ that $(w, y) \in R$ . So $w \in [y]$.

If $w \in [y]$ then $(w, y) \in R$.

Together with $(y, x) \in R$ (since $(x, y) \in R$ and $R$ is symmetric) it follows from the transitivity of $R$ that $(w, x) \in R$ . So $w \in [x]$.

Hence $[x] = [y]$.

"$\Leftarrow$": Suppose $[x] = [y]$.

Then, by (a), we have $x \in [y]$. So $(x, y) \in R$.

**(b)** $(x,y)\in R \quad\Leftrightarrow\quad [x]=[y]$

# Proof of (c)

**(c)** $[x]=[y]$ or $[x]\cap[y]=\varnothing$

**We will show that** $[x]\neq[y]$ **implies** $[x]\cap[y]=\varnothing$

**Proof by contradiction:**

**Suppose** $[x]\neq[y]$ **and suppose that** $[x]\cap[y]\neq\varnothing$

**Then there exists** $w\in A$ **with** $w\in[x]\cap[y]$

**Then:** $w\in[x]$ **and** $w\in[y]$ **So:** $(w,x)\in R$ **and** $(w,y)\in R$

**Then also** $(x,w)\in R$ **because** $R$ **is symmetric.**

**And** $(x,y)\in R$ **because** $R$ **is transitive.**

**But then (b) implies** $[x]=[y]$ **Contradiction !**

**Hence:** $[x]\cap[y]=\varnothing$

# Example

$A = Z$ **The relation *R* on *A* is given by:**

$(a, b) \in R$ **if *a* – *b* is divisible by 3.**

***R* is the “modulo 3 -relation on *Z* ”.**

**Verify that *R* is a equivalence relation.**

**This relation is often denoted by: $Z_3$ .**

**The three equivalence classes of $Z_3$ are:**

$[0]$ $[1]$ **and** $[2]$

# Theorem

Let *A* be a set. Then

(a) Each equivalence relation *R* on *A* defines a partition of *A*:
The set of equivalence classes of *R*.

(b) Each partition *P* of *A* defines an equivalence relation *R* on *A*:

$(x, y) \in R \quad \Leftrightarrow \quad x$ and $y$ are in the same block of $P$.

# Proof of (a)

**Each equivalence relation *R* on *A* defines a partition of *A*:**
**The set of equivalence classes of *R*.**

**Proof:**

**(1) We have** $x \in [x]$ **(by previous Theorem)**

**So** $\varnothing \neq [x] \subseteq A$

**(2) From** $x \in [x]$ **it follows that** $\bigcup_{x \in A} [x] = A$

**(3) From** $[x] = [y] \quad \text{or} \quad [x] \cap [y] = \varnothing$

**(cf. previous Theorem)**

**it follows that** $[x] \cap [y] = \varnothing$ **if** $[x] \neq [y]$

# Proof of (b)

**Each partition $P$ of $A$ defines an equivalence relation $R$ on $A$:**

$$(x, y)\in R \quad \Leftrightarrow \quad x \text{ and } y \text{ are in the same block of } P.$$

**Proof:**

**(1) $R$ is reflexive, since each $x\in A$ is in exactly one block of $P$.**

**(2) $R$ is obviously symmetric.**

**(3) $R$ is transitive because if $x$ and $y$ are in the same block of $P$ and $y$ and $z$ are in the same block of $P$, then so are $x$ and $z$.**

# Example 1

$$A = \{1,2,3,4,5\}$$

$$R = \{(1,1),(2,2),(3,3),(4,4),(5,5),(1,2),(2,1),(2,4),(4,2),(1,4),(4,1)\}$$

Verify that *R* is an equivalence relation.

Corresponding partition: $\{\{1,2,4\},\{3\},\{5\}\}$ $= \{[1],[3],[5]\}$

Conversely: if $A = \{1,2,3,4,5,6\}$

and $P = \{\{1\},\{2,5\},\{3,4,6\}\}$ is a partition of *A*

then *P* corresponds to the equivalence relation:

$$R = (\{1\}\times\{1\})\cup(\{2,5\}\times\{2,5\})\cup(\{3,4,6\}\times\{3,4,6\})$$

$$= \{(1,1),(2,2),(2,5),(5,2),(5,5),(3,3),(3,4),(3,6),(4,3),(4,4),(4,6),(6,3),(6,4),(6,6)\}$$

# Example 2

*A* is the set of all 0-1-strings of length 10.

The relation *R* on *A* is given by:

$(x, y) \in R$ if *x* and *y* have the same number of 1's.

a. Show that *R* is an equivalence relation.

b. Examine if *R* is antisymmetric.

c. Determine the partition of *A* induced by *R*.

# Solution

$(x, y) \in R$ if $x$ and $y$ have the same number of 1's.

a. Show that $R$ is an equivalence relation.

1. $R$ is reflexive, since $(x, x) \in R$ for all $x \in A$ :
   obviously the strings $x$ and $x$ have the same number of 1's.
2. $R$ is symmetric because
   if the strings $x$ and $y$ have the same number of 1's, then so have $y$ and $x$.
3. $R$ is transitive because
   if the strings $x$ and $y$ have the same number of 1's, and also $y$ and $z$ have the same number of 1's, then so have $x$ and $z$.

Hence $R$ is an equivalence relation.

$(x, y) \in R$ if $x$ and $y$ have the same number of 1's.

# Solution (continued)

b. Examine if *R* is antisymmetric

*R* is not antisymmetric, because, for example,

$(1000000000, 0000000001) \in R$ and also

$(0000000001, 1000000000) \in R$

but: $1000000000 \neq 0000000001$.

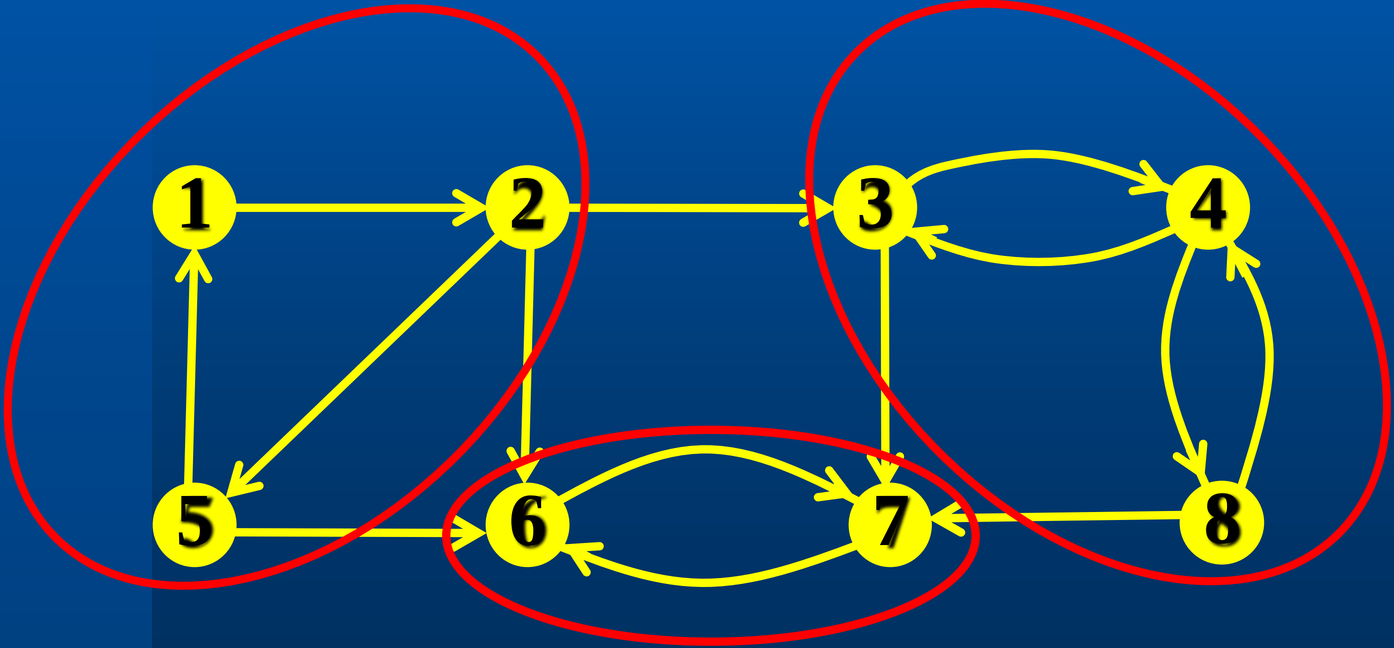
c. Determine the partition of *A* induced by *R*.

Solution: $\{A_0, A_1, A_2, \cdots, A_{10}\}$

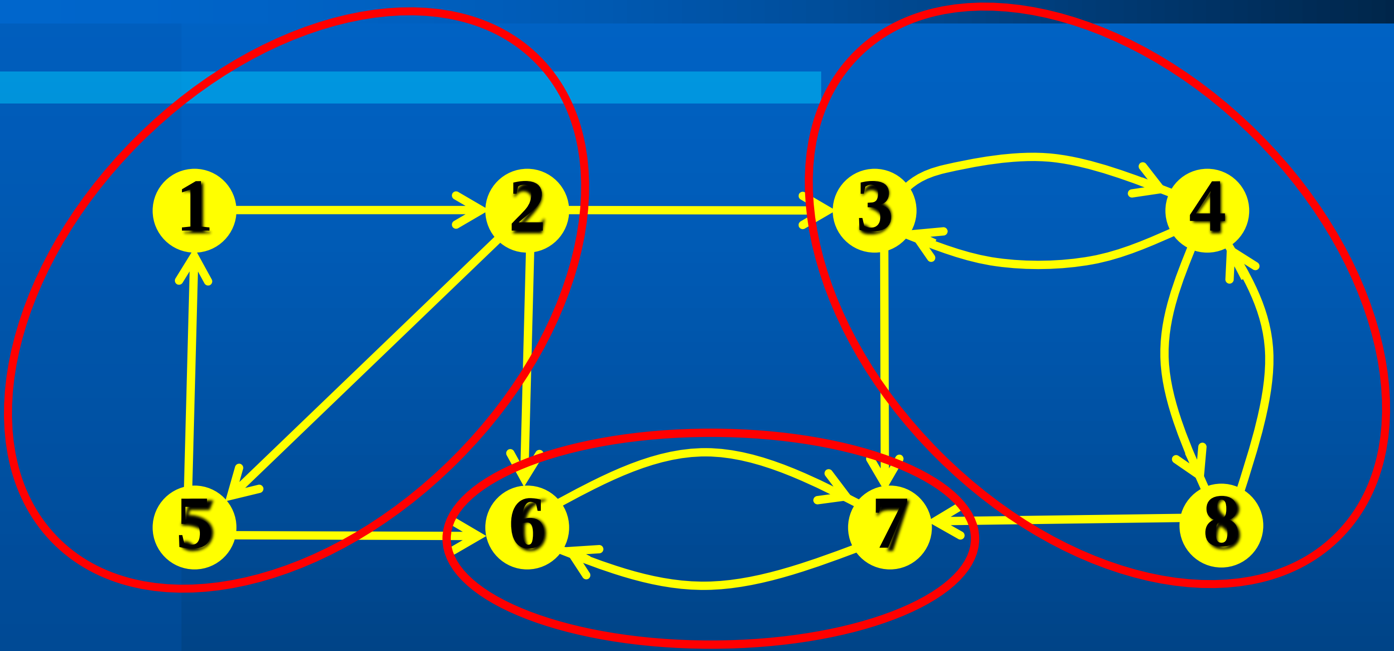
where $A_k$ denotes the set of strings with exactly $k$ ones $(0 \leq k \leq 10)$.

# Example 3

A <u>**strongly connected component**</u> of a directed graph *G* is a subgraph of *G* in which any two nodes are connected by a directed path.



# Example 3



**Theorem**:

**Each directed graph *G* has a unique partition of its node set into strongly connected components.**

# Example 3

**Proof**:

Strong connectedness is an equivalence relation *R* on the node set of *G*:

*xRy* if there is a directed path in *G* from *x* to *y* as well as a directed path from *y* to *x*.

Verify that this is indeed an equivalence relation (for reflexivity, consider a path of length 0).

Hence, the result follows from the previous theorem, since the equivalence classes of *R* correspond to the strongly connected components of *G*.