# Discrete Mathematics for Computer Science

## Lecture 12

# §7.2: Computer Recognition of Relations

**Definition:**

Let $A$, $B$ and $C$ be sets and let

$R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$ be relations.

Then the **Composite Relation** $R_1 \circ R_2 \subseteq A \times C$

is defined by:

$$R_1 \circ R_2 = \{(a,c) \in A \times C \mid \exists b \in B \left[(a,b) \in R_1 \wedge (b,c) \in R_2\right]\}$$

**Example:** $A = \{1,2,3,4\}$ $B = \{x, y, z\}$ $C = \{p, q, r\}$

$$R_1 = \{(1, y), (1, z), (2, y), (3, x), (4, z)\}$$

$$R_2 = \{(x, r), (y, p), (y, r), (z, p), (z, q)\}$$

**Then:** $R_1 \circ R_2 = \{(1, p), (1, r), (1, q), (2, p), (2, r), (3, r), (4, p), (4, q)\}$

# Theorem

Let $A$, $B$, $C$ and $D$ be sets and let

$R_1 \subseteq A \times B$ , $R_2 \subseteq B \times C$ and $R_3 \subseteq C \times D$ be relations.

Then: $R_1 \circ (R_2 \circ R_3) = (R_1 \circ R_2) \circ R_3$

**Proof:**

(i) We first show that $R_1 \circ (R_2 \circ R_3) \subseteq (R_1 \circ R_2) \circ R_3$

Let $(a,d) \in R_1 \circ (R_2 \circ R_3)$

Then $\exists b \in B \left[ (a,b) \in R_1 \wedge (b,d) \in R_2 \circ R_3 \right]$

$(b,d) \in R_2 \circ R_3$ implies $\exists c \in C \left[ (b,c) \in R_2 \wedge (c,d) \in R_3 \right]$

Hence $(a,c) \in R_1 \circ R_2$ and $(c,d) \in R_3$

And so: $(a,d) \in (R_1 \circ R_2) \circ R_3$

(ii) The proof of "$\supseteq$" is similar.

# Powers of a relation

**Let $R \subseteq A \times A$ be a relation on $A$.**

**Then:** $R^n = R \circ R \circ \cdots \circ R$

**Example:** $A = \{1,2,3,4\}$

$R = \{(1,2),(1,4),(2,3),(3,4),(4,2)\}$

**Then:** $R^2 = \{(1,2),(1,3),(2,4),(3,2),(4,3)\}$

$R^3 = \{(1,3),(1,4),(2,2),(3,3),(4,4)\}$ $R^4 = R$

**Note that if $|A|$ is finite, we must have $R^i = R^j$ for some $j > i$.**

# The matrix of a relation

**Let *A* and *B* be finite ordered sets, and let *R* be a relation from *A* to *B*.**

**Write** $A = \{a_1, a_2, \cdots, a_m\}$ $B = \{b_1, b_2, \cdots, b_n\}$

**The <u>Relation Matrix</u> *M*(*R*) associated with *R* is the *m*×*n*-(0,1)-matrix given by:**

$$M(R)_{ij} = \begin{cases} 1 & \text{if} \quad (a_i, b_j) \in R \\ 0 & \text{if} \quad (a_i, b_j) \notin R \end{cases}$$

**<u>Example</u>:**

**If** $A = \{1,2,3,4\}$ $B = \{x, y, z\}$

$R = \{(1, y), (1, z), (2, y), (3, x), (4, z)\}$

**Then** $M(R) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

# Theorem

**If $A$, $B$ and $C$ are finite ordered sets and $R_1 \subseteq A\times B$ and $R_2 \subseteq B\times C$ relations, then**

$$M(R_1 \circ R_2) = M(R_1)\cdot M(R_2)$$

**where in the matrix product on the right hand side the boolean addition $1 + 1 = 1$ is used.**

$R_1 \subseteq A \times B$ $R_2 \subseteq B \times C$ $M(R_1 \circ R_2) = M(R_1) \cdot M(R_2)$

**Example** $A = \{1,2,3,4\}$ $B = \{x, y, z\}$ $C = \{p, q, r\}$

$$R_1 = \{(1, y), (1, z), (2, y), (3, x), (4, z)\}$$

$$R_2 = \{(x, r), (y, p), (y, r), (z, p), (z, q)\}$$

**Then**

$$R_1 \circ R_2 = \{(1, p), (1, r), (1, q), (2, p), (2, r), (3, r), (4, p), (4, q)\}$$

**We have:**

$$M(R_1) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M(R_2) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad M(R_1 \circ R_2) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

**Verify that indeed:** $M(R_1 \circ R_2) = M(R_1) \cdot M(R_2)$

# Proof

$R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$

$$M(R_1 \circ R_2) = M(R_1) \cdot M(R_2)$$

**Write** $A = \{a_1, a_2, \cdots, a_m\}$ $B = \{b_1, b_2, \cdots, b_n\}$ $C = \{c_1, c_2, \cdots, c_k\}$

**Using the dot-product definition of matrix-multiplication (Math C1),**

**we get for $1 \le i \le m$ and $1 \le j \le k$ :** $(M(R_1) \cdot M(R_2))_{ij} = 1$

$\Leftrightarrow$ $(i^{th} \text{ row of } M(R_1)) \bullet (j^{th} \text{ column of } M(R_2)) = 1$

**1 + 1 = 1**

$\Leftrightarrow$ $\sum_{t=1}^{n} \left[ M(R_1)_{it} \cdot M(R_2)_{tj} \right] = 1$ $\Leftrightarrow$ $\exists_{t \in \{1,\ldots,n\}} \left[ M(R_1)_{it} \cdot M(R_2)_{tj} = 1 \right]$

$\Leftrightarrow$ $\exists_{t \in \{1,\ldots,n\}} \left[ M(R_1)_{it} = 1 \wedge M(R_2)_{tj} = 1 \right]$

$\Leftrightarrow$ $\exists_{t \in \{1,\ldots,n\}} \left[ (a_i, b_t) \in R_1 \wedge (b_t, c_j) \in R_2 \right]$ $\Leftrightarrow$ $(a_i, c_j) \in R_1 \circ R_2$

$\Leftrightarrow$ $M(R_1 \circ R_2)_{ij} = 1$

# Theorem

Let $A$ a finite ordered set and let $R \subseteq A \times A$ be a relation on $A$. Write $n = |A|$.

Then:

(i) $R$ is reflexive if and only if $I_n \leq M(R)$

Here $I_n$ denotes the $n \times n$ identity-matrix and "≤" means that each element in $I_n$ is less than or equal to the corresponding element in $M(R)$.

(ii) $R$ is symmetric if and only if $M(R) = M(R)^T$

Here $M(R)^T$ denotes the transpose of $M(R)$.

# Theorem

**Let $A$ a finite ordered set and let $R \subseteq A \times A$ be a relation on $A$. Write $n = |A|$.**

**Then:**

**(iii) $R$ is transitive if and only if** $\left(M(R)\right)^2 \leq M(R)$

**(iv) $R$ is antisymmetric if and only if**

$$M(R) \cap \left(M(R)\right)^T \leq I_n$$

**where** $\left[M(R) \cap \left(M(R)\right)^T\right]_{ij} = M(R)_{ij} \cdot M(R)_{ij}^T$

# Proof of (iii)

(iii) *R* is transitive if and only if $(M(R))^2 \le M(R)$

"⇒" Suppose *R* is transitive. Write $A = \{a_1, a_2, \cdots, a_m\}$

We must show that if $(M(R))^2_{ij} = 1$ for some $1 \le i, j \le n$ then also $M(R)_{ij} = 1$

Well: $(M(R))^2_{ij} = 1$ means $\sum_{t=1}^{n} \left[ M(R)_{it} \cdot M(R)_{tj} \right] = 1$

So $\exists_{t \in \{1,\ldots,n\}} \left[ M(R)_{it} = 1 \wedge M(R)_{tj} = 1 \right]$

So $\exists_{t \in \{1,\ldots,n\}} \left[ (a_i, a_t) \in R \wedge (a_t, a_j) \in R \right]$

Now the transitivity of *R* implies that $(a_i, a_j) \in R$

And so: $M(R)_{ij} = 1$

# Proof of (iii)

$A = \{a_1, a_2, \cdots, a_m\}$

(iii) *R* is transitive if and only if $(M(R))^2 \leq M(R)$

"⇐" Suppose $(M(R))^2 \leq M(R)$

We must show that *R* is transitive.

Well: let $1 \leq i, s, j \leq n$, and suppose

$(a_i, a_s) \in R \wedge (a_s, a_j) \in R$ (we will show that $(a_i, a_j) \in R$)

We have: $M(R)_{is} = 1$ and $M(R)_{sj} = 1$

And so $(M(R))^2_{ij} = \sum_{t=1}^{n} \left[ M(R)_{it} \cdot M(R)_{tj} \right] = 1$

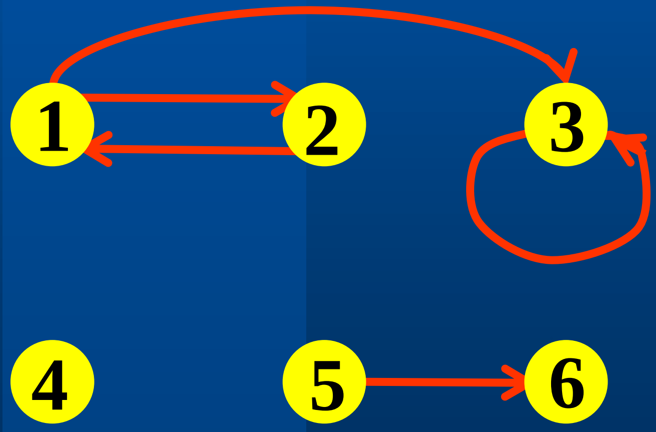
Then, by $(M(R))^2 \leq M(R)$ we have $M(R)_{ij} = 1$

Hence: $(a_i, a_j) \in R$

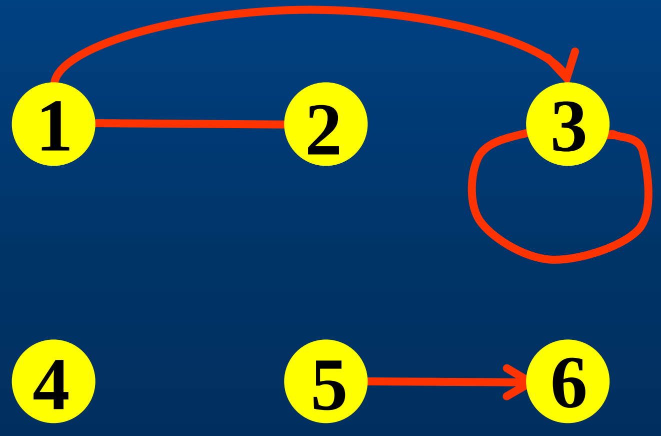
# The Directed Graph of a Relation

**The directed graph corresponding to a relation *R* on a finite set *A* is the directed graph with vertex set *A* and arc set *R*.**

**Example:** $A = \{1,2,3,4,5,6\}$ $R = \{(1,2),(1,3),(2,1),(3,3),(5,6)\}$




**If both $(x, y) \in R$ and $(y, x) \in R$, the corresponding arcs can be replaced by an undirected edge:**



# Properties of Directed Graph of a Relation

1. *R* is reflexive if and only if there is a loop at each vertex.
2. *R* is symmetric if and only if the graph can be drawn merely using undirected edges.
3. *R* is transitive if and only if the graph has the property that whenever there is a directed path from $x$ to $y$, also the arc $(x, y)$ is present.
4. *R* is antisymmetric if and only if the graph does not have any undirected edges, except for possible loops $(x, x)$.

# §7.3: Partial Orders

**Definition:**

A relation $R$ on $A$ is called a **partial order** if:

$R$ is reflexive, antisymmetric and transitive.

If $R$ is a partial order on $A$, then $(A, R)$ is called a **partial ordered set**, or **poset**.

**Examples:**

$(Z, \leq)$ and $(Z, \geq)$ are posets.

For a set $X$, the "$\subseteq$–relation on $P(X)$" is a partial order. So $(P(X), \subseteq)$ is a poset.

# PERT networks

Another example of posets are PERT (Program Evaluation and Review Technique) Networks.

E.g: $A$ = {jobs to be done to build a house}

$R = \{(i, j) \mid$ job $i$ must precede job $j\}$

(Cf. instructions manuals)

# Directed Networks of Posets

Note that the directed graph associated with a poset has the following properties:

1. There is a loop at each vertex (reflexivity)
2. If there is a directed path from $x$ to $y$, also the arc $(x, y)$ is present (transitivity).
3. There are no directed cycles in the graph, aside from loops (transitivity + antisymmetry).

# Hasse Diagrams

The directed graph associated to a poset is often drawn as a Hasse Diagram:

1. All loops are removed
2. Edges are drawn undirected and are assumed to be directed from bottom to top.
3. If $(x, y)$ is an edge, there is no vertex $z$ such that both $(x, z)$ and $(z, y)$ are also edges (transitivity is assumed, but not drawn)

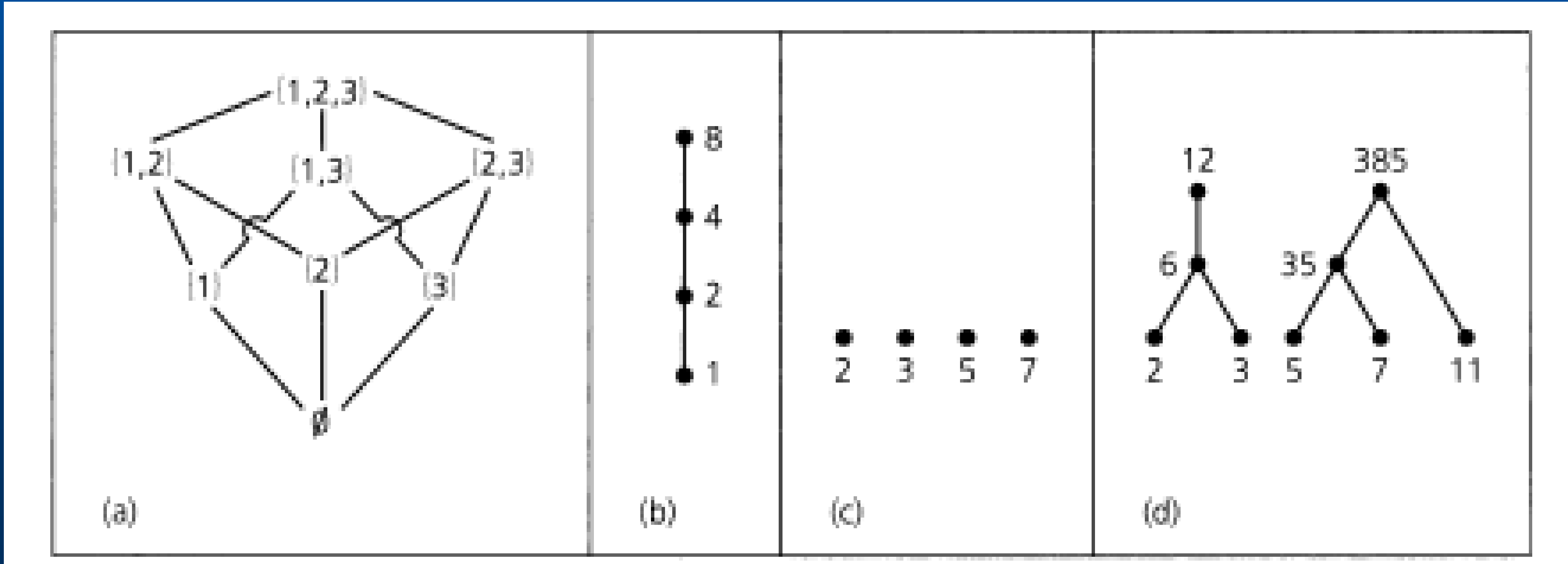
# Hasse Diagrams: Examples

(a) The ($P(\{1,2,3\}), \subseteq$)-poset

(b) The posets ($A$, $R$), where $A = \{1, 2, 4, 8\}$ and
$R = \{(x,y) \mid x \text{ divides } y\}$

(c) $A = \{2, 3, 5, 7\}$ and $R$ as in (b)

(d) $A = \{2, 3, 5, 6, 7, 11, 12, 35, 385\}$ and $R$ as in (b)



# Totally Ordered Posets

A poset $(A, R)$ is called Totally Ordered if for all $x, y \in A$, either $(x, y) \in R$ or $(y, x) \in R$.

Examples:

1. The poset (b) on the previous slide.
2. The posets $(Z, \leq)$ and $(Z, \geq)$.

Note: the Hasse diagram of a totally ordered poset consists of a single (vertical) chain.
So a totally ordered poset orders all elements of $A$.

# Topological Sorting Algoritm

Let $(A, R)$ be a poset. Write $|A| = n$.

The Topological Sorting Algoritm constructs a totally ordered poset $(A, T)$, such that $R \subseteq T$.

It therefore arranges the $n$ elements of $A$ in such a way that it satisfies the partial order $R$.

**Step 1**: Set $k = 1$. Let $H_1$ be the Hasse diagram of $R$.

**Step 2**: Select a vertex $v_k$ in $H_k$ such that no edge starts at $v_k$ ($v_k$ is a top-vertex)

**Step 3**:

If $k = n$ we have the total order $v_k < v_{k-1} < \cdots < v_1$

If $k < n$ remove $v_k$ and its adjacent edges from $H_k$ yielding $H_{k+1}$ ; put $k := k+1$ and return to Step 2.

# Topological Sorting Algoritm

**<u>Example</u>: Let ($A$, $R$) be the poset given by:**

**$A$ = {A, B, C, D, E, F, G} and**

**$R$ = {((A,C), (B,A), (C,D), (C,F), (C,G), (E,A)}**

**The following figure demonstrates the intermediate phases of the Topological Sorting Algoritm.**