Discrete Mathematics for Computer Science

Lecture 11

§5.1: Cartesian Products and Relations

Definition:

A (binary) relation from set A to set B is a subset of $A \times B$.

Example: **If** $A = \{1, 2, x\}$ **and** $B = \{2, y\}$

Then Relations from A to B are (among others):

$\{(1, y)\}$ $\{(1,2), (2,2), (2, y)\}$ $\varnothing$ $A \times B$

§7.1: Relations (2nd round)

Definition:

A (binary) relation from a set A to the (same) set A is a subset of $A \times A$.

Such a relation is called a relation on A.

Example 1: $A = \{1, 2, x\}$

Relations on A are (among others):

$\{(1, x)\}$ $\{(1,2), (2,2), (2, x)\}$ $\varnothing$ $A \times A$

Notation:

If $R = \{(1,2), (2,2), (2,x)\}$, then another notation for $(1,2) \in R$ is: $1R2$.

Relations on *A*

Example 2: $A = Z$

The relation *R* on *A* is given by:

$(a, b) \in R$ **if** $a \le b$

***R* is the "$\le$-relation on *Z*".**

E.g: $(1,2) \in R$ $(-7,-3) \in R$ $(4,4) \in R$

$(-1,-2) \notin R$ $(1,0) \notin R$

Relations on *A*

Example 3: $A = Z$

The relation *R* on *A* is given by:

$(a, b) \in R$ **if *a* – *b* is divisible by 11.**

***R* is the "modulo 11 -relation on *Z* ".**

E.g: $(13,2) \in R$ $(2,-9) \in R$ $(0,11) \in R$ $(3,25) \in R$

$(11,1) \notin R$ $(2,9) \notin R$

Relations on *A*

Example 4: Consider a universe U.

Let $C \subseteq U$. The relation R on $P(U)$ is given by:

$(A, B) \in R$ if $A \cap C = B \cap C$

E.g: if $U = \{1,2,3,4,5,6\}$ and $C = \{1,3,4\}$

Then: $(\{1,2,3\}, \{1,3,5,6\}) \in R$ since: $\{1,2,3\} \cap C = \{1,3\}$

and also $\{1,3,5,6\} \cap C = \{1,3\}$

But: $(\{2,3\}, \{3,4,6\}) \notin R$ since: $\{2,3\} \cap C = \{3\}$

and: $\{3,4,6\} \cap C = \{3,4\}$

Reflexivity

Definition:

A relation R on A is reflexive if:

$(a,a) \in R$ **for all** $a \in A$

Example 1: $A = \{1,2,x\}$

$R = \{(1,1),(2,2),(x,x),(2,x)\}$ **is reflexive**

$R = \{(1,1),(x,x),(2,x),(2,1),(x,1)\}$ **is not reflexive**

since $(2,2) \notin R$

$(a,a) \in R$ **for all** $a \in A$

Reflexivity

Example 2:

The " $\leq$ **-relation on** Z **" is reflexive, because**

$a \leq a$ **for all** $a \in Z$

Note: the "< -relation on Z**" is not reflexive!**

Example 3:

The modulo 11 -relation on Z **is reflexive because:**

$a - a$ **(= 0) is divisible by 11, for all** $a \in Z$**.**

$(a, a) \in R$ **for all** $a \in A$

Reflexivity

Example 4: **Consider a universe** U**.**

Let $C \subseteq U$**. The relation** R **on** $P(U)$ **is given by:**

$(A, B) \in R$ **if** $A \cap C = B \cap C$

R **is reflexive because** $(A, A) \in R$ **for all** $A \in P(U)$**,**

Since $A \cap C = A \cap C$ **for all** $A \in P(U)$**.**

Symmetry

Definition:

A relation R on A is symmetric if:

$(a,b) \in R \Rightarrow (b,a) \in R$ **for all** $a,b \in A$

Example 1: $A = \{1,2,x\}$

$R = \{(1,1),(2,2),(x,x),(2,x)\}$ **is not symmetric**

since $(2,x) \in R$ **but** $(x,2) \notin R$

$R = \{(1,2),(x,x),(2,x),(2,1),(x,2)\}$ **is symmetric**

$(a,b) \in R \Rightarrow (b,a) \in R$ **for all** $a,b \in A$

Symmetry

Example 2:

The " $\leq$ -relation on *Z* " is not symmetric,

E.g: $2 \leq 3$ **but not** $3 \leq 2$

Example 3:

The modulo 11 -relation on *Z* is symmetric, because:

if $a - b$ **is divisible by 11, then so is** $b - a$**:**

(since: $b - a = -(a - b)$**)**

$(a,b) \in R \Rightarrow (b,a) \in R$ **for all** $a,b \in A$

Symmetry

Example 4: **Consider a universe *U*.**

Let $C \subseteq U$. The relation *R* on *P*(*U*) is given by:

$(A,B) \in R$ **if** $A \cap C = B \cap C$

R **is symmetric, because for all** $A,B \in P(U)$**:**

if $A \cap C = B \cap C$ **then also** $B \cap C = A \cap C$

Transitivity

Definition:

A relation R on A is transitive if:

$$\left((a,b)\in R \wedge (b,c)\in R\right) \Rightarrow (a,c)\in R$$ **for all** $a,b,c\in A$

Example 1: $A=\{1,2,x,y\}$

$R=\{(1,2),(x,x)\}$ **is transitive**

$R=\{(1,2),(y,2),(1,x),(2,x),(y,x)\}$ **is transitive**

$R=\{(2,1),(2,2),(y,x),(1,x)\}$ **is not transitive**

since $(2,1),(1,x)\in R$ **but** $(2,x)\notin R$

$\left((a,b) \in R \land (b,c) \in R\right) \Rightarrow (a,c) \in R$ **for all** $a,b,c \in A$

Transitivity

Example 2:

The " $\leq$ -relation on *Z* " is transitive,

since: $\left(a \leq b \land b \leq c\right) \Rightarrow a \leq c$

Example 3:

The modulo 11 -relation on *Z* is transitive, because:

if both $a - b$ and $b - c$ are divisible by 11,

then so is $a - c$, since: $a - c = (a - b) + (b - c)$

$((a,b) \in R \wedge (b,c) \in R) \Rightarrow (a,c) \in R$ **for all** $a,b,c \in A$

Transitivity

Example 4: **Consider a universe *U*.**

Let $C \subseteq U$. The relation *R* on *P*(*U*) is given by:

$(A,B) \in R$ **if** $A \cap C = B \cap C$

R **is transitive,**
because for all *A*, *B*, *D*∈*P(U*):

if $A \cap C = B \cap C$ **and** $B \cap C = D \cap C$

then also $A \cap C = D \cap C$

Antisymmetry

Definition:

A relation R on A is antisymmetric if:

$\big((a,b)\in R \wedge (b,a)\in R\big) \Rightarrow a=b$ **for all** $a,b\in A$

Example 1: $A=\{1,2,x\}$

$R=\{(x,1),(2,2),(1,x),(2,x)\}$ **is not antisymmetric**

since $(x,1),(1,x)\in R$ **but** $1\neq x$

$R=\{(1,2),(x,x),(2,x),(2,2)\}$ **is antisymmetric**

$$\left((a,b)\in R \wedge (b,a)\in R\right) \Rightarrow a=b$$ **for all** $a,b\in A$

Antisymmetry

Example 2:

The " $\leq$ **-relation on** Z **" is antisymmetric,**

since: $\left(a\leq b \wedge b\leq a\right) \Rightarrow a=b$

Example 3:

The modulo 11 -relation on Z **is not antisymmetric, since:**

12 – 1 is divisible by 11, and so is 1 – 12.

But: 1 $\neq$ **12 !**

$\left((a,b)\in R \wedge (b,a)\in R\right)\Rightarrow a=b$ **for all** $a,b\in A$

Antisymmetry

Example 4: **Consider a universe *U*.**

Let $C\subseteq U$. The relation *R* on *P*(*U*) is given by:

$(A,B)\in R$ **if** $A\cap C = B\cap C$

***R* is not antisymmetric .**

E.g: take $U=\{1,2,3,4,5,6\}$ **and** $C=\{1,3,4\}$

Then: $(\{1,2,3\},\{1,3,5,6\})\in R$ **since:** $\{1,2,3\}\cap C=\{1,3\}$

and: $\{1,3,5,6\}\cap C=\{1,3\}$

Also: $(\{1,3,5,6\},\{1,2,3\})\in R$ **(same reason)**

But: $\{1,2,3\}\neq\{1,3,5,6\}$

Antisymmetry

Remark:

A relation R on A can be symmetric as well as antisymmetric:

Example: $A = \{1,2\}$ **and** $R = \{(1,1),(2,2)\}$

A relation R on A can be neither symmetric nor antisymmetric:

Example: $A = \{1,2,3\}$ **and** $R = \{(1,2),(2,1),(2,3)\}$

Example

Let R_1 and R_2 be relations on a set A.

Prove or disprove (using a counterexample) the following statements:

(a) If R_1 and R_2 are both reflexive, then so is $R_1 - R_2$

(b) If R_1 and R_2 are both symmetric, then so is $R_1 - R_2$.

(c) If R_1 and R_2 are both transitive, then so is $R_1 - R_2$.

(d) If R_1 and R_2 are both antisymmetric, then so is $R_1 - R_2$.

Solution

(a) If R_1 and R_2 are both reflexive, then so is $R_1 - R_2$

False!

E.g: $A = \{1,2\}$ **and** $R_1 = R_2 = \{(1,1),(2,2)\}$

Then R_1 and R_2 are reflexive, but $R_1 - R_2 = \varnothing$ **is not.**

(b) If R_1 and R_2 are symmetric, then so is $R_1 - R_2$.

Proof: Let $(x, y) \in R_1 - R_2$.

Then $(x, y) \in R_1$ and $(x, y) \notin R_2$.

Then $(y, x) \in R_1$, since R_1 is symmetric.

And $(y, x) \notin R_2$, because $(y, x) \in R_2$ would imply $(x, y) \in R_2$, since R_2 is symmetric.

So $(y, x) \in R_1 - R_2$.

Solution (continued)

(c) If R_1 and R_2 are transitive then so is $R_1 - R_2$.

False!

E.g: $A = \{1,2,3\}$ $R_1 = \{(1,2),(2,3),(1,3)\}$ $R_2 = \{(1,3)\}$

Then R_1 and R_2 are transitive,

but $R_1 - R_2 = \{(1,2),(2,3)\}$ **is not.**

(d) If R_1 and R_2 are antisymmetric, then so is $R_1 - R_2$.

Proof: Suppose $(x, y) \in R_1 - R_2$ and $(y, x) \in R_1 - R_2$

Then $(x, y) \in R_1$ and $(y, x) \in R_1$.

So $x = y$, because R_1 is antisymmetric.

Partial order

Definition:

A relation R on A is called a partial order if:

R is reflexive, antisymmetric and transitive.

Examples:

For a set X, the "$\subseteq$ -relation on $P(X)$" is a partial order.

The modulo 11 -relation on Z is not a partial order (since it is not antisymmetric)

Equivalence Relations

Definition:

A relation *R* on *A* is called an **equivalence relation** if:

R is reflexive, symmetric and transitive.

Examples:

The modulo 11 -relation on *Z* is an equivalence relation.

The " $\leq$ -relation on *Z* " is not an equivalence relation (since it is not symmetric).