Discrete Mathematics for Computer Science

Lecture 10

§5.4: Special functions

Definitions:

A function f from $A \times A$ to B is called a binary operation on A. If $B \subseteq A$ then f is closed on A.

A function f from A to A is called a unary or monary operation on A.

Examples:

$f: R \times R \to R$ defined by $f(a,b) = a - b$ is a closed binary operation on R ("*subtraction*")

$f: Z^{+} \times Z^{+} \to Z$ defined by $f(a,b) = a - b$ is not closed

$f: R^{+} \to R^{+}$ defined by $f(x) = \frac{1}{x}$ is a unary operation on R^{+} ("*inversion*")

Commutative and Associative Operations

Definitions:

Let $f : A \times A \to B$ be a binary operation on *A*.

f is called commutative if

$$f(x, y) = f(y, x) \text{ for all } x, y \in A.$$

If *f* is closed, then is called associative if

$$f(f(x, y), z) = f(x, f(y, z)) \text{ for all } x, y, z \in A.$$

Examples:

Addition and multiplication are commutative and associative operations on *R*.

Subtraction is not commutative, since $x - y \neq y - x$

and neither associative, since $(x - y) - z \neq x - (y - z)$

Commutative and Associative Operations

Example:

Let *U* be a universe.

Define $f : P(U) \times P(U) \to P(U)$ **by** $f(A, B) = A \cap B$

Then *f* is a commutative binary operation on *P*(*U*) since, for all *A*, *B*∈*P*(*U*):

$$f(A, B) = A \cap B = B \cap A = f(B, A)$$

***f* is also associative, since for all *A*, *B*, *C*∈*P*(*U*):**

$$f\left(f(A, B), C\right) = (A \cap B) \cap C$$

$$= A \cap \left(B \cap C\right) = f\left(A, f\left(B, C\right)\right)$$

Identity Element

Definition:

Let $f: A \times A \to B$ be a binary operation on A.

$x \in A$ is an identity (element) for f if

$$f(x, a) = f(a, x) = a \text{ for all } a \in A.$$

Examples:

0 is an identity for the Addition-operation and 1 is an identity for the Multiplication-operation.

The Subtraction-operations does not have an identity, since $f(a, x) = a - x = a$ implies $x = 0$

but then $f(x, a) = x - a = 0 - a = -a \neq a$ (unless $a = 0$)

Identity Element

Examples:

Let *U* be a universe.

Define $f : P(U) \times P(U) \to P(U)$ **by** $f(A, B) = A \cap B$

Then *U* is an identity element of *f*.

The binary operation $g : P(U) \times P(U) \to P(U)$ **given by** $g(A, B) = A \cup B$ **has an identity** $\varnothing$**.**

Uniqueness of Identity Element

Theorem:

Let $f : A \times A \to B$ be a binary operation on *A*.

If *f* has an identity, then it is unique.

Proof:

We use rule of contradiction (From Lecture 2).

Let *x* and *y* be two distinct identities of *f*.

Since *y* is an identity, we have: $f(x, y) = x$

Since *x* is an identity, we have: $f(x, y) = y$

Therefore: *x* = *y*. Using initial assumption this creates a contradicition. So the identity is unique.

Projections

Definition:

Let A and B be sets and let $D \subseteq A \times B$.

The function $\pi_A : D \to A$ given by: $\pi_A(a,b) = a$
is called the projection of D on the first coordinate.

The function $\pi_B : D \to B$ given by: $\pi_{|B}(a,b) = b$

is called the projection of D on the 2nd coordinate.

Example:

Let $D \subseteq R \times R$ be given by: $D = \{(x,y) \mid (x-1)^2 + (y+2)^2 \leq 9\}$
(disk with radius 3 and center (1, –2))

Then $\pi_A(D) = [-2,4]$ and $\pi_B(D) = [-5,1]$

Projections (generalization)

Definition:

Let A_1, A_2, …, A_n be sets and let $\{i_1, i_2, \ldots, i_m\} \subseteq \{1, 2, \ldots, n\}$

with $m \le n$ and $i_1 < i_2 < \ldots < i_m$.

Let $D \subseteq A_1 \times A_2 \times \cdots \times A_n$

The function $\pi : D \to A_{i_1} \times A_{i_2} \times \cdots \times A_{i_m}$ **given by:**

$$\pi(a_1, a_2, \cdots, a_n) = (a_{i_1}, a_{i_2}, \cdots, a_{i_m})$$

is called the projection of D

on the i_1th, i_2th,…, i_mth coordinates.

Example: Relational Data Bases

A_1: Courses, A_2: Programs,
A_3: Teachers, A_4: Classrooms

A_1 = {Math A, Math B1, Math B2, Math C1}

A_2 = {CS, BIT, TBK}

A_3 = {A, B, C, D, E, F}

A_4 = {SP3, SP4, SP5, WA3, WA4}

Example: Relational Data Bases

Consider $D \subseteq A_1 \times A_2 \times A_3 \times A_4$ given in the following table:

Course	Program	Teacher	Classroom
Math A	CS	A	SP3
Math B1	CS	B	WA4
Math B2	CS	D	SP5
Math C1	CS	C	WA4
Math A	BIT	A	SP3
Math B1	BIT	E	SP3
Math B2	BIT	D	SP5
Math C1	BIT	F	WA3
Math A	TBK	A	SP3
Math B1	TBK	A	SP4
Math B2	TBK	D	SP3
Math C1	TBK	C	WA4

Example: Relational Data Bases

Projection $\pi(D)$ of D on 1st, 3rd and 4th coordinates is given by:

Course	Teacher	Classroom
Math A	A	SP3
Math B1	B	WA4
Math B2	D	SP5
Math C1	C	WA4
Math B1	E	SP3
Math C1	F	WA3
Math B1	A	SP4
Math B2	D	SP3

Projection $\pi(D)$ of D on 2nd and 4th coordinates is given by:

Program	Classroom
CS	SP3
CS	WA4
CS	SP5
BIT	SP3
BIT	SP5
BIT	WA3
TBK	SP3
TBK	SP4
TBK	WA4

§5.6: Function Composition

Definition:

If $f: A \to B$ and $g: B \to C$ then the **composite function** $g \circ f: A \to C$ is given by:

$$(g \circ f)(a) = g(f(a)) \text{ for each } a \in A.$$

Examples: $A = \{1,2,4,5\}$ $B = \{a,b,d\}$ $C = \{w,x,y,z\}$

$f = \{(1,b),(2,a),(4,d),(5,b)\}$ $g = \{(a,x),(b,z),(d,w)\}$

Then: $(g \circ f) = \{(1,z),(2,x),(4,w),(5,z)\}$

Note: $f \circ g$ is not defined here!

If $A = B = C = R$ and $f(x) = x^3$ $g(x) = x+1$

Then: $(g \circ f)(x) = x^3 + 1$ and $(f \circ g)(x) = (x+1)^3$

Theorem

Let $f : A \to B$ and $g : B \to C$.

Then:

(a) If f and g are both one-to-one then also $g \circ f$ is one-to-one.

(b) If f and g are both onto then also $g \circ f$ is onto.

Equality of functions

Definition:

Let $f: A \to B$ and $g: A \to B$ then

$f = g$ if $f(a) = g(a)$ for all $a \in A$.

Example: $f: R \to Z$ and $g: R \to Z$ given by:

$$f(x) = \begin{cases} x & \text{if} \quad x \in Z \\ \lceil x \rceil - 1 & \text{if} \quad x \notin Z \end{cases}$$

$$g(x) = \lfloor x \rfloor$$

Theorem

Let $f : A \to B$, $g : B \to C$ and $h : C \to D$

Then: $$(h \circ g) \circ f = h \circ (g \circ f)$$

So function composition is an associative operation (but not a commutative operation!)

<u>Remark</u>: another notation for $f \circ f$ is f^2

In general: $$f^n = f \circ f \circ \cdots \circ f$$

Inverse of a function

Definition:

Let $f : A \to B$ then *f* is called invertible if there exists a function $g : B \to A$ such that

$$g \circ f = 1_A \quad \textbf{and} \quad f \circ g = 1_B$$

where for a set *C* the function $1_C : C \to C$ **denotes the *identity function* given by:**

$$1_C(x) = x \quad \text{for all } x \in C$$

Theorem

Let $f : A \to B$ and suppose a function $g : B \to A$ satisfies

$$g \circ f = 1_A \quad \text{and} \quad f \circ g = 1_B$$

Then g is unique.

Proof: **Suppose function $h : B \to A$ also satisfies**

$$h \circ f = 1_A \quad \text{and} \quad f \circ h = 1_B$$

We will show that $h = g$, i.e, $h(x) = g(x)$ for all $x \in B$.

$$h(x) = h(1_B(x)) = (h \circ 1_B)(x) = (h \circ (f \circ g))(x)$$

$$= ((h \circ f) \circ g)(x) = (1_A \circ g)(x) = 1_A(g(x)) = g(x)$$

Thm. 5.6

Remark: g is called the inverse of f and is denoted by: f^{-1}

Examples

$A=\{1,2,3,4\}$ $B=\{a,b,c,d\}$

If $f=\{(1,b),(2,d),(3,c),(4,a)\}$ **then** $f^{-1}=\{(a,4),(b,1),(c,3),(d,2)\}$

$A=B=[0,\infty)$

If $f(x)=x^2$ **then** $f^{-1}(x)=\sqrt{x}$

$A=B=(0,\infty)$

If $f(x)=\frac{1}{x}$ **then** $f^{-1}(x)=\frac{1}{x}$

$A=(1,2]$ $B=[-1,1)$

If $f(x)=-2x+3$ **then** $f^{-1}(x)=\frac{-x+3}{2}$

Theorem

Let $f : A \to B$. Then f is invertible
if and only if
f is both one-to-one and onto.

Remark:
A function that is both one-to-one and onto is called **bijective** or a **one-to-one correspondence**.

Theorem

Let $f: A \to B$, $g: B \to C$ be invertible functions.

Then: $g \circ f$ is also invertible and

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

Proof: Exercise 5.6.11.

Theorem

Let $f : A \to B$ with $|A| = |B|$.
Then the following statements are equivalent:

(a) f is one-to-one.

(b) f is onto.

(c) f is invertible.

Preimage

Definition: Let $f : A \to B$ and $B_1 \subseteq B$. Then

$$f^{-1}(B_1) = \{x \in A \mid f(x) \in B_1\}$$

$f^{-1}(B_1)$ is called the Preimage of B_1 under f.

Example: If $A = \{1,2,4,5\}$ $B = \{a, b, d\}$

and $f = \{(1,b),(2,a),(4,d),(5,b)\}$

Then: $f^{-1}(\{b,d\}) = \{1,4,5\}$

Remark: Note that $f^{-1}(B_1)$ is always defined, even when f is not invertible !!

$$f^{-1}(B_1)=\{x\in A \mid f(x)\in B_1\}$$

Theorem

Let $f: A \to B$ and $B_1, B_2 \subseteq B$. Then

$$(a)\quad f^{-1}(B_1\cap B_2)=f^{-1}(B_1)\cap f^{-1}(B_2)$$

$$(b)\quad f^{-1}(B_1\cup B_2)=f^{-1}(B_1)\cup f^{-1}(B_2)$$

$$(c)\quad f^{-1}\left(\overline{B_1}\right)=\overline{f^{-1}(B_1)}$$

Proof of (c): $x\in f^{-1}\left(\overline{B_1}\right) \Leftrightarrow f(x)\in \overline{B_1} \Leftrightarrow f(x)\notin B_1$

$\Leftrightarrow x\notin f^{-1}(B_1) \Leftrightarrow x\in \overline{f^{-1}(B_1)}$