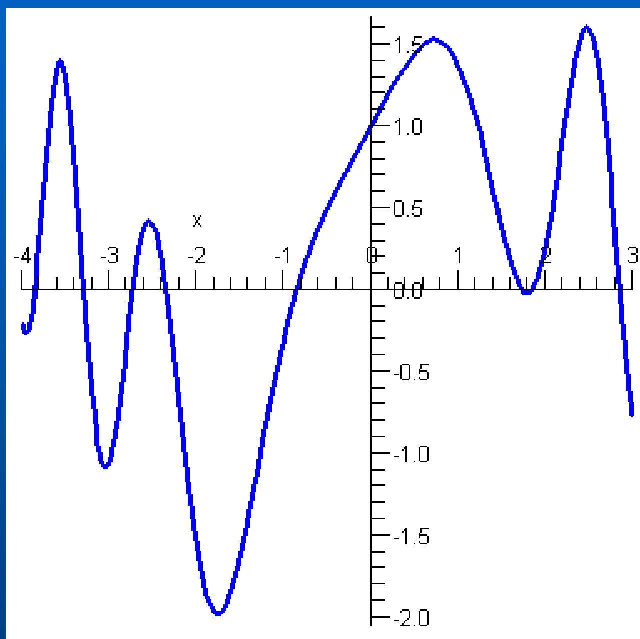
Discrete Mathematics for Computer Science

Lecture 9

Chapter 5: Relations and Functions



Function $f(x) = \sin x + \cos x^2$

Relation

§5.1: Cartesian Products and Relations

René Descartes
(1595-1650)

Definition:

The Cartesian Product of sets *A* and *B* is defined by: $A \times B = \{(a,b) \mid a \in A, b \in B\}$

Examples: **If** $A = \{1,2,x\}$ **and** $B = \{2,y\}$

Then: $A \times B = \{(1,2),(1,y),(2,2),(2,y),(x,2),(x,y)\}$

$B \times A = \{(2,1),(2,2),(2,x),(y,1),(y,2),(y,x)\}$

$B^2 = B \times B = \{(2,2),(2,y),(y,2),(y,y)\}$

Note:

$|A \times B| = |A| \cdot |B|$ **So:** $|A \times B| = |B \times A|$ **but** $A \times B \neq B \times A$

Cartesian Products and Relations

Definition:

The Cartesian Product of sets *A*, *B* and *C* is defined by:

$$A \times B \times C = \{(a,b,c) \mid a \in A, b \in B, c \in C\}$$ **etc.**

Examples: **If** $A = \{1,2,x\}$ $B = \{2,y\}$ $C = \{1,3\}$

Then: $C \times A \times B$

$$= \{(1,1,2),(1,1,y),(1,2,2),(1,2,y),(1,x,2),(1,x,y),$$
$$(3,1,2),(3,1,y),(3,2,2),(3,2,y),(3,x,2),(3,x,y)\}$$

$$C^3 = C \times C \times C$$

$$= \{(1,1,1),(1,1,3),(1,3,1),(1,3,3),(3,1,1),(3,1,3),(3,3,1),(3,3,3)\}$$

Cartesian Products and Relations

Definition:

A (binary) relation from set *A* to set *B* is a subset of *A*×*B*.

Example: **If** $A = \{1, 2, x\}$ **and** $B = \{2, y\}$

Then Relations from *A* to *B* are (among others):

$\{(1, y)\}$ $\{(1,2), (2,2), (2, y)\}$ $\varnothing$ $A \times B$

The number of relations from *A* to *B* is equal to:

$$|P(A \times B)| = 2^{|A \times B|} = 2^{|A||B|}$$

Cartesian Products and Relations

Example:

$A = \{1,2,3,4\}$ $B = \{2,4,5\}$

The "≥" relation from *A* to *B* is given by:

$$\{(2,2),(3,2),(4,2),(4,4)\}$$

Cartesian Products and Relations

Theorem:

Let *A*, *B* , and *C* be sets in a universe *U*.

Then:

$$(a) \quad A\times\left(B\cap C\right)=\left(A\times B\right)\cap\left(A\times C\right)$$

$$(b) \quad A\times\left(B\cup C\right)=\left(A\times B\right)\cup\left(A\times C\right)$$

$$(c) \quad \left(A\cap B\right)\times C=\left(A\times C\right)\cap\left(B\times C\right)$$

$$(d) \quad \left(A\cup B\right)\times C=\left(A\times C\right)\cup\left(B\times C\right)$$

Cartesian Products and Relations

Proof of (b): $(b)\quad A\times(B\cup C)=(A\times B)\cup(A\times C)$

Part 1: $A\times(B\cup C)\subseteq(A\times B)\cup(A\times C)$

Let $(x,y)\in A\times(B\cup C)$

Then $x\in A$ **and** $y\in B\cup C$ **(Def. "×")**

So $x\in A$ **and** ($y\in B$ **or** $y\in C$)

So ($x\in A$ **and** $y\in B$) **or** ($x\in A$ **and** $y\in C$)

So $(x,y)\in A\times B$ **or** $(x,y)\in A\times C$ **(Def. "×")**

Thus $(x,y)\in(A\times B)\cup(A\times C)$

Part 2: $(A\times B)\cup(A\times C)\subseteq A\times(B\cup C)$ **Reverse the steps in Part 1**

§5.2: Functions

Definition: (A and B are nonempty sets)
A **function** f from A to B is a relation R from A to B, such that each element of A appears exactly once as the first component in R.

Notation: $f: A \to B$

Examples: If $A = \{1,2,4\}$ $B = \{2,3,5,6\}$ Then:

$\{(1,1),(2,3),(4,5)\}$ is not a function from A to B.

$\{(1,5),(2,3),(4,3)\}$ is a function from A to B.

$\{(1,2),(4,3)\}$ is not a function from A to B.

$\{(1,2),(2,3),(2,6),(4,3)\}$ is not a function from A to B.

Functions $f : A \to B$

Notation:

Instead of $(a, b) \in f$, we mostly write $f(a) = b$.

- b is the **image** of a (under f).
- a is a **pre-image** of b (under f).
- A is the **domain** of f.
- B is the **codomain** of f.
- $f(A) = \left\{ f(a) \middle| a \in A \right\}$ is the **range** of f.

Note that: if $f(a) = b$ and $f(a) = c$ then necessarily $b = c$

Examples of Functions (1)

Let $f: R \to R$

E.g: $f(x) = x^2$ $f(x) = \sin x$ $f(x) = e^x$

The floor function: $f: R \to Z$ $f(x) \to \lfloor x \rfloor$

$\lfloor x \rfloor$ is the largest integer $\leq x$.

So: $\lfloor 4 \rfloor = 4$ $\lfloor 2.7 \rfloor = 2$ $\lfloor -2.2 \rfloor = -3$

Examples of Functions (2)

The <u>ceiling</u> function: $f : R \to Z$ $f(x) \to \lceil x \rceil$

$\lceil x \rceil$ **is the smallest integer** $\geq x$**.**

So:

$\lceil 4 \rceil = 4$ $\lceil 2.3 \rceil = 3$ $\lceil -2.9 \rceil = -2$

Examples of Functions (3)

Let $m,n \in Z^+$ and let A be a $m \times n$-matrix.

The access function:

$$f : \left(\left\{a_{ij} \mid 1 \le i \le m; 1 \le j \le n\right\}\right) \to \left\{1, 2, \ldots, mn\right\}$$

given by $f(a_{ij}) = (i-1)n + j$

stores the entries of matrix A row by row in a one dimensional array.

($f(a_{ij})$ denotes the location of a_{ij} in this array)

Number of Functions $f: A \rightarrow B$

For each element of *A* one can choose among |*B*| images, so the number of functions from *A* to *B* is equal to:

$$|B|^{|A|}$$

Note:

The number of functions from *A* to *B* is in general not equal to the number of functions from *B* to *A*.

Functions: One-to-One

Definition:

A function f from A to B is one-to-one (injective) if each element of B appears at most once as the image of an element of A.

Example: If $A = \{1,2,4\}$ and $B = \{2,3,5,6\}$ then

$\{(1,6),(2,3),(4,5)\}$ is one-to one.

$\{(1,6),(2,3),(4,3)\}$ is not one-to one.

$f : R \to Z$ given by $f(x) \to \lfloor x \rfloor$ is not one-to one.

Note that a function $f : A \to B$ is one-to-one if and only if $\forall x \in A \forall y \in A [f(x) = f(y) \to x = y]$

Functions: One-to-One

If $f: A \rightarrow B$ is one-to-one, then necessarily: $|A| \leq |B|$

If $|A| = m$ and $|B| = n$ ($m \leq n$), then the number of one-to-one functions from A to B is equal to:
(write $A = \{a_1, a_2, \cdots\cdots, a_m\}$)

$$n \cdot (n-1) \cdot (n-2) \cdots\cdots (n-m+1) = \frac{n!}{(n-m)!} = P(|B|, |A|)$$

(Math A)

choices for $f(a_1)$ choices for $f(a_2)$ choices for $f(a_m)$

Functions $f : A \to B$

Definition: Let $f : A \to B$ and $A_1 \subseteq A$.

The *image of* A_1 under f is:

$$f(A_1) = \{ b \in B \mid b = f(a) \text{ for some } a \in A_1 \}$$

Examples:

If $A = \{1,2,3,4\}$ $B = \{2,3,5,6,7\}$ $A_1 = \{1,2,4\}$

and $f = \{(1,5),(2,3),(3,6),(4,5)\}$

Then: $f(A_1) = \{3,5\}$

If $A = R$, $B = Z$, $A_1 = (-3.5\ ,\ 1.5]$ and $f(x) = \lceil x \rceil$

Then: $f(A_1) = \{-3,-2,-1,0,1,2\}$

Functions $f : A \to B$

Theorem: Let $f : A \to B$ and $A_1 , A_2 \subseteq A$.

Then:

$$(a) \quad f\left(A_1 \cup A_2\right) = f\left(A_1\right) \cup f\left(A_2\right)$$

$$(b) \quad f\left(A_1 \cap A_2\right) \subseteq f\left(A_1\right) \cap f\left(A_2\right)$$

$$(c) \quad f\left(A_1 \cap A_2\right) = f\left(A_1\right) \cap f\left(A_2\right)$$ **if f is one-to-one**

Proof of $(c) \quad f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$

if *f* is one-to-one

Let $f : A \to B$ and $A_1, A_2 \subseteq A$ and assume that *f* is one-to-one.

(i) First we show that: $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$

Let: $b \in f(A_1 \cap A_2)$

Then there exists $a \in A_1 \cap A_2$ with $f(a) = b$

Then $a \in A_1$ and $a \in A_2$ (Def "$\cap$")

So $b \in f(A_1)$ and $b \in f(A_2)$

Hence: $b \in f(A_1) \cap f(A_2)$

Proof of $(c) \quad f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$

if *f* is one-to-one

(ii) Now we show that $f(A_1) \cap f(A_2) \subseteq f(A_1 \cap A_2)$

Let: $b \in f(A_1) \cap f(A_2)$

Then $b \in f(A_1)$ **and** $b \in f(A_2)$

So, there exists $x \in A_1$ **and** $y \in A_2$

with $f(x) = b$ **and** $f(y) = b$

Since *f* is one-to one, we `must have *x* = *y* !!

So $x \in A_1$ **and** $x \in A_2$

So, there exists $x \in A_1 \cap A_2$ **with** $f(x) = b$

Hence: $b \in f(A_1 \cap A_2)$

Restrictions and Extensions

Definition: Let $f : A \to B$ and $A_1 \subseteq A$.

The restriction $f\big|_{A_1} : A_1 \to B$ of f to A_1 is given by:

$$f\big|_{A_1}(a) = f(a) \text{ for all } a \in A_1$$

Definition: Let $A_1 \subseteq A$ and $f : A_1 \to B$

A function $g : A \to B$ is an extension of f to A if:

$$g(a) = f(a) \text{ for all } a \in A_1$$

Restrictions and Extensions

Examples:

If $A = \{1,2,4\}$ $B = \{2,3,5,6\}$ $A_1 = \{1,4\}$

and $f = \{(1,5),(2,3),(4,5)\}$

Then: $f\big|_{A_1} = \{(1,5),(4,5)\}$ **is the restriction of *f* to A_1.**

If $h : A_1 \rightarrow B$ is given by: $h = \{(1,6),(4,3)\}$

Then $g = \{(1,6),(4,3),(2,2)\}$ **is an extension of *h* to *A*.**

§5.3: Onto-functions

Definition:

A function f from A to B is onto (surjectief) if each element of B appears as an image.

In other words: if $f(A) = B$.

Examples: If $A = \{1,2,4,5\}$ $B = \{1,3,4\}$ **Then:**

$\{(1,4),(2,4),(4,1),(5,3)\}$ **is onto.**

$\{(1,4),(2,4),(4,1),(5,4)\}$ **is not onto.**

$f : R \to Z$ **given by** $f(x) \to \lfloor x \rfloor$ **is onto.**