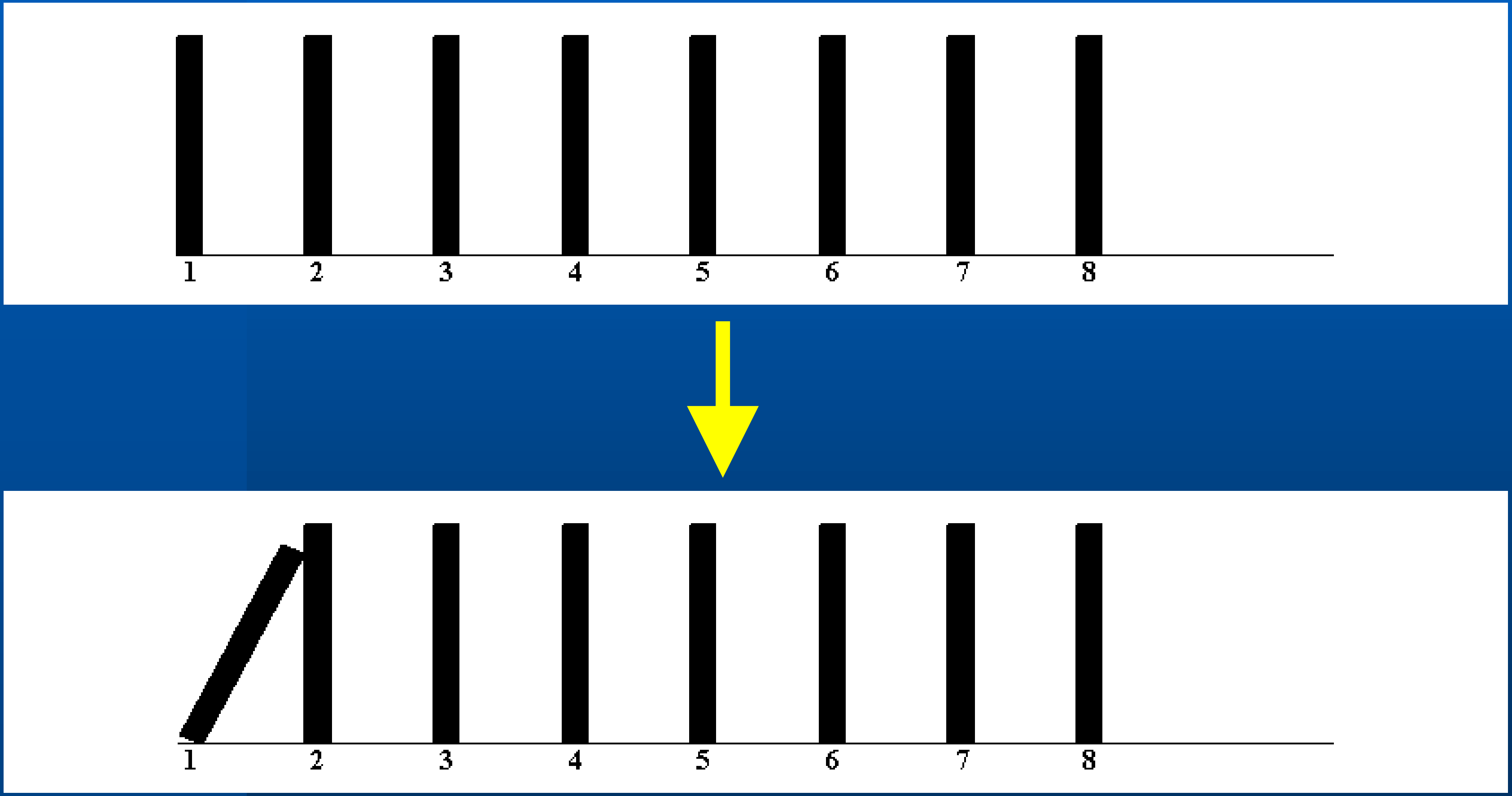
Discrete Mathematics for Computer Science

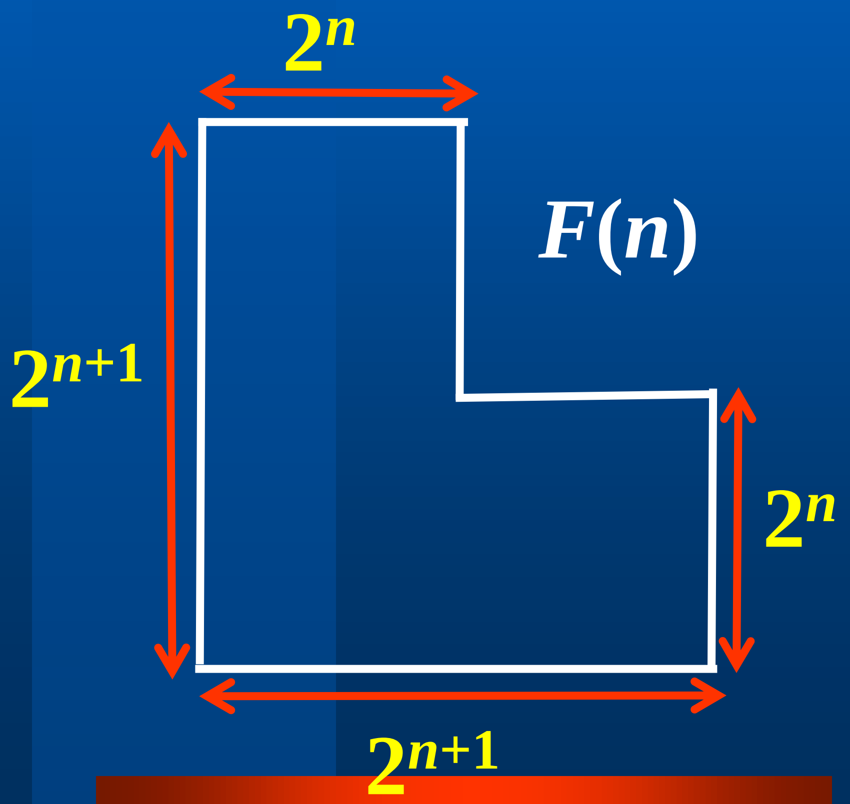
Lecture 8

Mathematical Induction

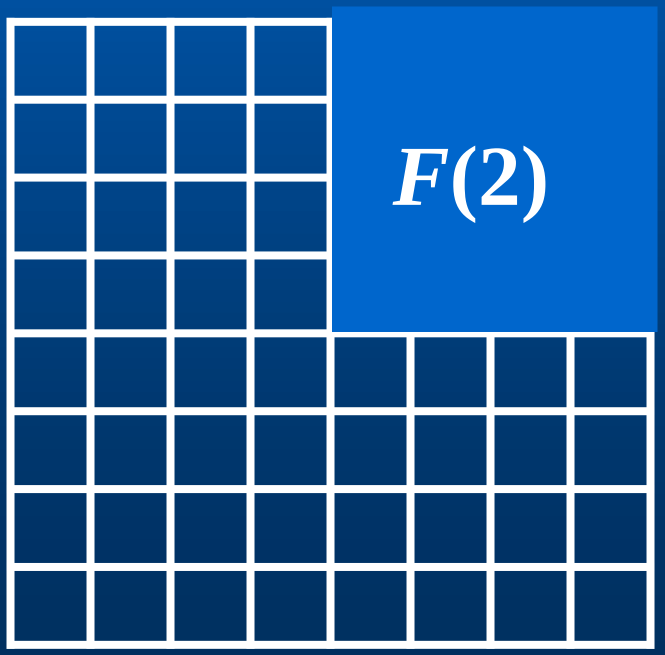


Example 1

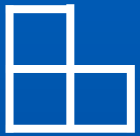
Consider for $n \in N$ the following figure consisting of 1×1 squares. We denote this figure with $F(n)$.

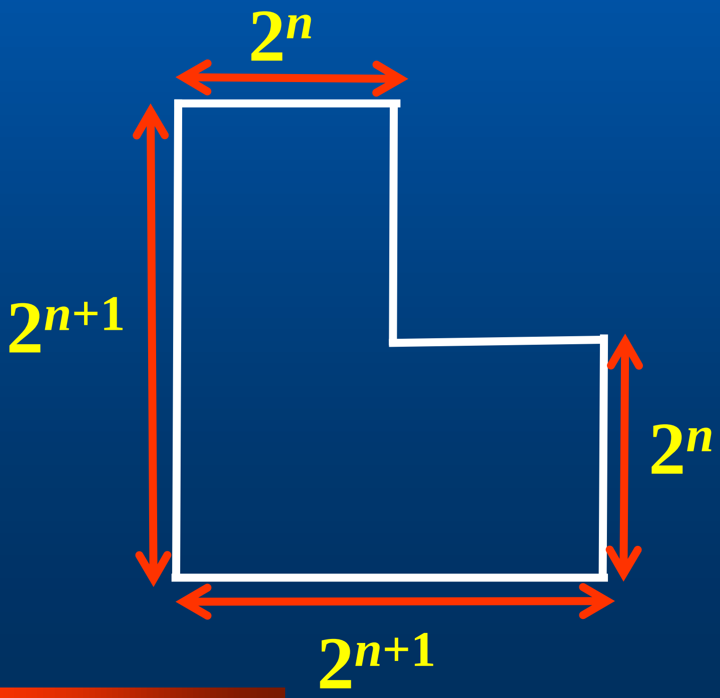


E.g, $n = 2$:



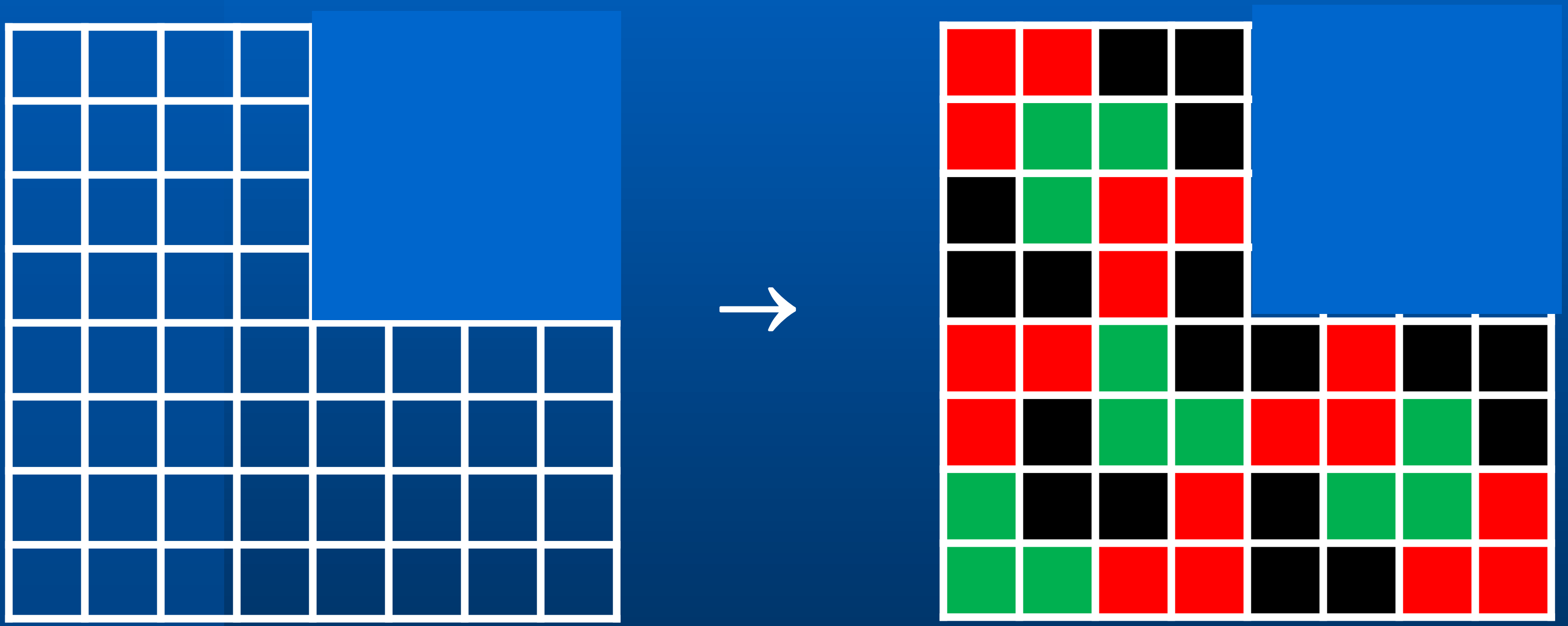
Example 1

Prove that for each $n \in N$, $F(n)$ can be covered with  -shaped tiles (corresponding to $F(0)$). Rotation of these tiles is allowed.



Example 1

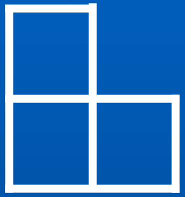
A solution for *n* = 2:



Proof

Mathematical Induction to *n*.

1. **Basis Step** for $n = 0$:



This is obvious, since then the figure is itself a tile.

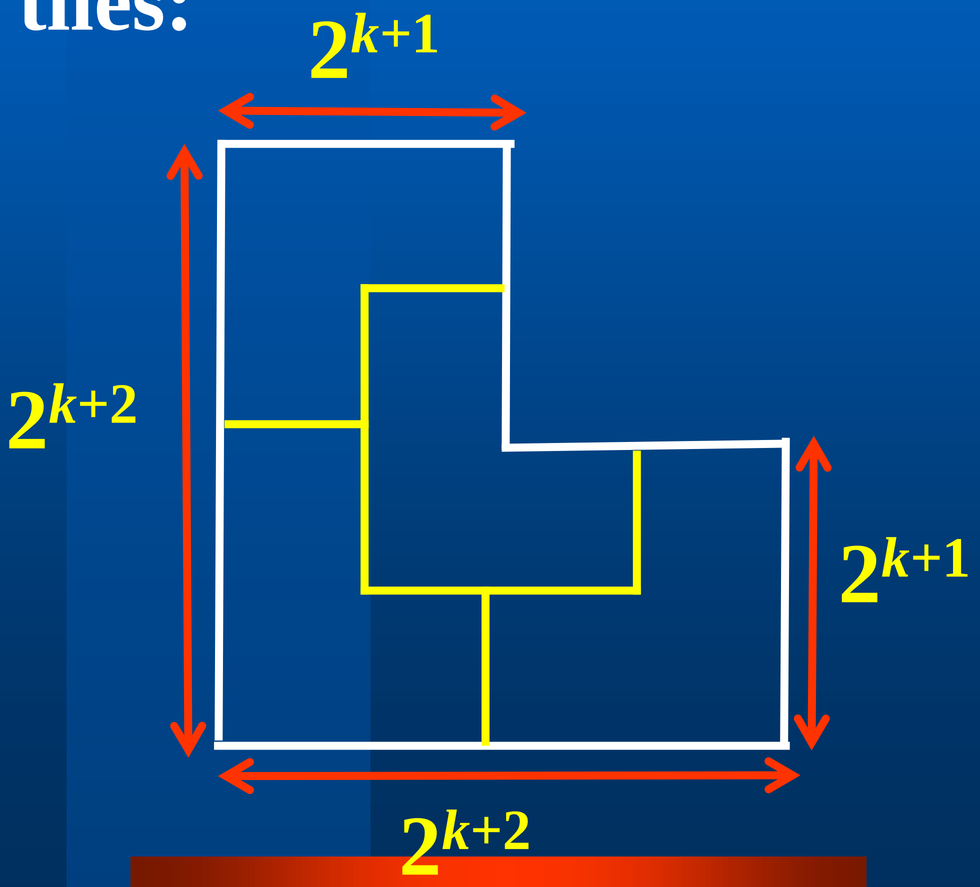
2. **Induction Step**:

Let $k \geq 0$ and suppose that $F(k)$ can be covered with $F(0)$ -shaped tiles. **IH**

We will show that then also $F(k+1)$ can be covered with $F(0)$ -shaped tiles.

Proof (continued)

Well: *F*(*k*+1) can be tiled with four *F*(*k*)-shaped tiles:

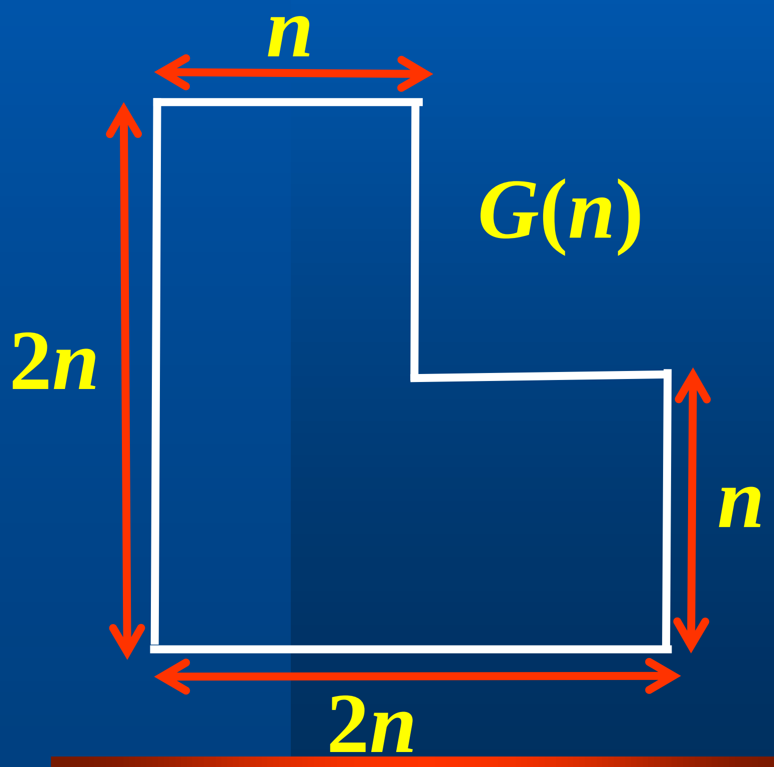


By IH, all these *F*(*k*)'s can be tiled with *F*(0)-shaped tiles.

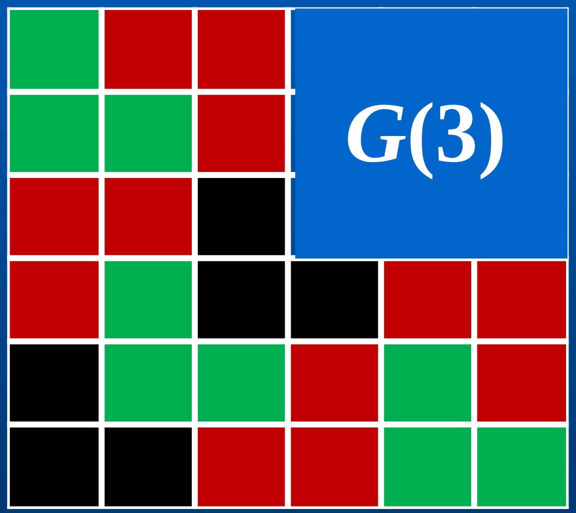
Therefore also *F*(*k*+1) can be tiled with *F*(0)-shaped tiles.

Challenge

The assertion of Example 1 even holds for all figures $G(n)$, $n \geq 1$ (consisting of $3n^2$ 1×1- squares)!



E.g: $n = 3$:



Can you prove this?

Example 2

Prove with Mathematical Induction that each *n* ≥ 60 can be written as sum of 7's and 11's; i.e, there exist *p*, *q* ∈*N* such that $n = 7p + 11q$

Proof:

1. Basis Step:

The statement is true for *n* = 60, because

$60 = 7 \cdot 7 + 11 \cdot 1$ **(take *p* = 7 and *q* = 1)**

Proof (continued)

2. **Induction Step**:

Let $k \geq 60$ and suppose that there exist $p, q \in N$ such that $k = 7p + 11q$

IH

We will show that then exist $s, t \in N$ such that $k + 1 = 7s + 11t$

We will distinguish the cases $q \geq 5$ and $q \leq 4$.

If $q \geq 5$ then, by IH $k + 1 = 7p + 11q + 1$

$$= 7p + 11(q - 5) + 11 \cdot 5 + 1$$

$$= 7p + 11(q - 5) + 56$$

$$= 7(p + 8) + 11(q - 5)$$

So take

$s = p + 8$

$t = q - 5$

Proof (continued)

2. **Induction Step**:

Let $k \geq 60$ and suppose that there exist $p, q \in N$ such that $k = 7p + 11q$

IH

We will show that then there exist $s, t \in N$ such that $k+1 = 7s + 11t$

We will distinguish the cases $q \geq 5$ and $q \leq 4$.

If $q \leq 4$, then $p \geq 3$ (because $k \geq 60$).

So then, by IH: $k+1 = 7p + 11q + 1$

$= 7(p-3) + 7 \cdot 3 + 11q + 1$

$= 7(p-3) + 11q + 22$ $= 7(p-3) + 11(q+2)$

So take

$s = p - 3$

$t = q + 2$

Example 3 (Introduction to Th. 4.2)

Let $a_1 = 1$; $a_2 = 2$ and $a_3 = 3$.

For $n \geq 4$, a_n is (recursively) defined by:

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$

So: $a_4 = a_3 + a_2 + a_1 = 3 + 2 + 1 = 6$

$a_5 = a_4 + a_3 + a_2 = 6 + 3 + 2 = 11$ **etc.**

Prove that for all $n \in Z^+$: $a_n \leq 2^n$

$a_1 = 1$ $a_2 = 2$ $a_3 = 3$ $a_n = a_{n-1} + a_{n-2} + a_{n-3} \quad \text{if } n \geq 4$

Proof

1. **Basis Step**:

The statement is true for *n* = 1, because

$$a_1 = 1 \leq 2^1$$

2. **Induction Step**:

Let $k \geq 1$ and suppose that $a_k \leq 2^k$ **IH**

We will show that: $a_{k+1} \leq 2^{k+1}$

$a_1 = 1$ $a_2 = 2$ $a_3 = 3$ $a_n = a_{n-1} + a_{n-2} + a_{n-3} \quad \text{if } n \geq 4$

Proof (continued)

$a_k \leq 2^k$ **IH**

We will show that: $a_{k+1} \leq 2^{k+1}$

IH

Well: $a_{k+1} = a_k + a_{k-1} + a_{k-2}$ $\leq 2^k + a_{k-1} + a_{k-2}$

Now we cannot proceed because we only assumed the Induction Hypothesis for *k* !
(and not for *k*–1 and *k*–2)

Therefore we introduce the strong version of Mathematical Induction (Theorem 4.2)

Mathematical induction(strong version)

Theorem 4.2

Let, for each $n \in Z^+$, $S(n)$ be an open statement.

(so the truth value of $S(n)$ depends on n)

Then $S(n)$ is true for all $n \in Z^+$ if:

1. $S(1)$, $S(2)$, …, $S(p)$ are all true (for some $p \geq 1$) and

2. For all $k \geq p$:

$$\left(S(1) \wedge S(2) \wedge \cdots\cdots \wedge S(k)\right) \rightarrow S(k+1)$$

(1. is the Basis Step, 2. is the Induction Step)

Remarks to Theorem 4.2

1. **The difference with Theorem 4.1 is that there might be more work to be done in the Basis Step, but that in the Induction Step a stronger Induction Hypothesis can be applied.**

2. **This strong Principle of Mathematical Induction not only holds for Z^+, but for each set of the form: $\{n_0, n_0+1, n_0+2, n_0+3, \cdots\cdots\}$ with $n_0 \in Z$**

Example 3 (second attempt)

$a_1 = 1$ $a_2 = 2$ $a_3 = 3$ $a_n = a_{n-1} + a_{n-2} + a_{n-3} \quad \text{if } n \geq 4$

Prove that for all $n \in Z^+$: $a_n \leq 2^n$

In our first attempt we have seen that we need the Induction Hypothesis for *k* as well as *k*–1 and *k*–2. (the <u>three</u> preceding statements of *S*(*k*+1)).

Therefore we apply Theorem 4.2, with *p* = 3.

Show that: $a_n \leq 2^n$

Proof

$a_1 = 1$ $a_2 = 2$ $a_3 = 3$ $a_n = a_{n-1} + a_{n-2} + a_{n-3} \quad \text{if} \;\; n \geq 4$

Proof:

1. **Basis Step**:

- **$S(1)$ is true, since :** $a_1 = 1$ **and** $2^1 = 2$
- **$S(2)$ is true, since :** $a_2 = 2$ **and** $2^2 = 4$
- **$S(3)$ is true, since :** $a_3 = 3$ **and** $2^3 = 8$

Proof (continued)

2. **Induction Step**:

Let $k \geq 3$ and suppose that

$a_1 \leq 2^1$ $a_2 \leq 2^2$ ……. $a_{k-2} \leq 2^{k-2}$ $a_{k-1} \leq 2^{k-1}$ $a_k \leq 2^k$ **IH**

We will show that: $a_{k+1} \leq 2^{k+1}$

Proof (continued)

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} \quad \text{if } n \geq 4$$

$a_1 \leq 2^1$ $a_2 \leq 2^2$ ……. $a_{k-2} \leq 2^{k-2}$ $a_{k-1} \leq 2^{k-1}$ $a_k \leq 2^k$ **IH** **($k \geq 3$)**

To prove: $a_{k+1} \leq 2^{k+1}$

IH

Well: $a_{k+1} = a_k + a_{k-1} + a_{k-2}$ $\leq 2^k + 2^{k-1} + 2^{k-2}$

Now it remains to show that: $2^k + 2^{k-1} + 2^{k-2} \leq 2^{k+1}$

Dividing both hands with 2^{k-2} **yields:**

$$2^2 + 2^1 + 1 \leq 2^3$$

Which is obviously true: $7 \leq 8$

Example 4 (Binary Search Algoritm)

Let, for $n \geq 1$, $a_1, a_2, a_3, \cdots, a_n$ be an increasing array of integers.

We use the following algoritm to determine whether a given integer j is a member of this array:

- If n is even: compare j with $a_{\frac{n}{2}}$
  - If $j = a_{\frac{n}{2}}$ then j is a member of the array
  - If $j < a_{\frac{n}{2}}$ continue searching in $a_1, \cdots, a_{\frac{n}{2}-1}$
  - If $j > a_{\frac{n}{2}}$ continue searching in $a_{\frac{n}{2}+1}, \cdots, a_n$
- If n is odd: compare j with $a_{\frac{n+1}{2}}$, etc.

Example 4 (Binary Search Algoritm)

Let *f*(*n*) denote the number of comparisons needed to determine whether a given integer *j* is a member of the increasing array: $a_1, a_2, a_3, \cdots, a_n$

Prove with Mathematical Induction that for all $n \geq 1$, $f(n) \leq 1 + \log_2 n$

Proof:

1. **Basis Step**:

The statement is true for $n = 1$, because:

$f(1) = 1$ and $1 + \log_2 1 = 1 + 0 = 1$

Proof (continued)

2. **Induction Step**:

Let $k \geq 1$ and suppose that for all $1 \leq p \leq k$

$$f(p) \leq 1 + \log_2 p$$

IH

We will show that: $f(k+1) \leq 1 + \log_2(k+1)$

Well, if k is odd, we deduce from the algoritm:

$$f(k+1) \leq 1 + f\left(\tfrac{k+1}{2}\right) \leq 1 + \left(1 + \log_2\left(\tfrac{k+1}{2}\right)\right)$$

$$= 1 + \left(1 + \log_2(k+1) - \log_2 2\right)$$

$$= 1 + \left(1 + \log_2(k+1) - 1\right)$$

$$= 1 + \log_2(k+1)$$

IH

($p = (k + 1)/2$)

If k is even, the proof of the induction step is analogous (verify!)