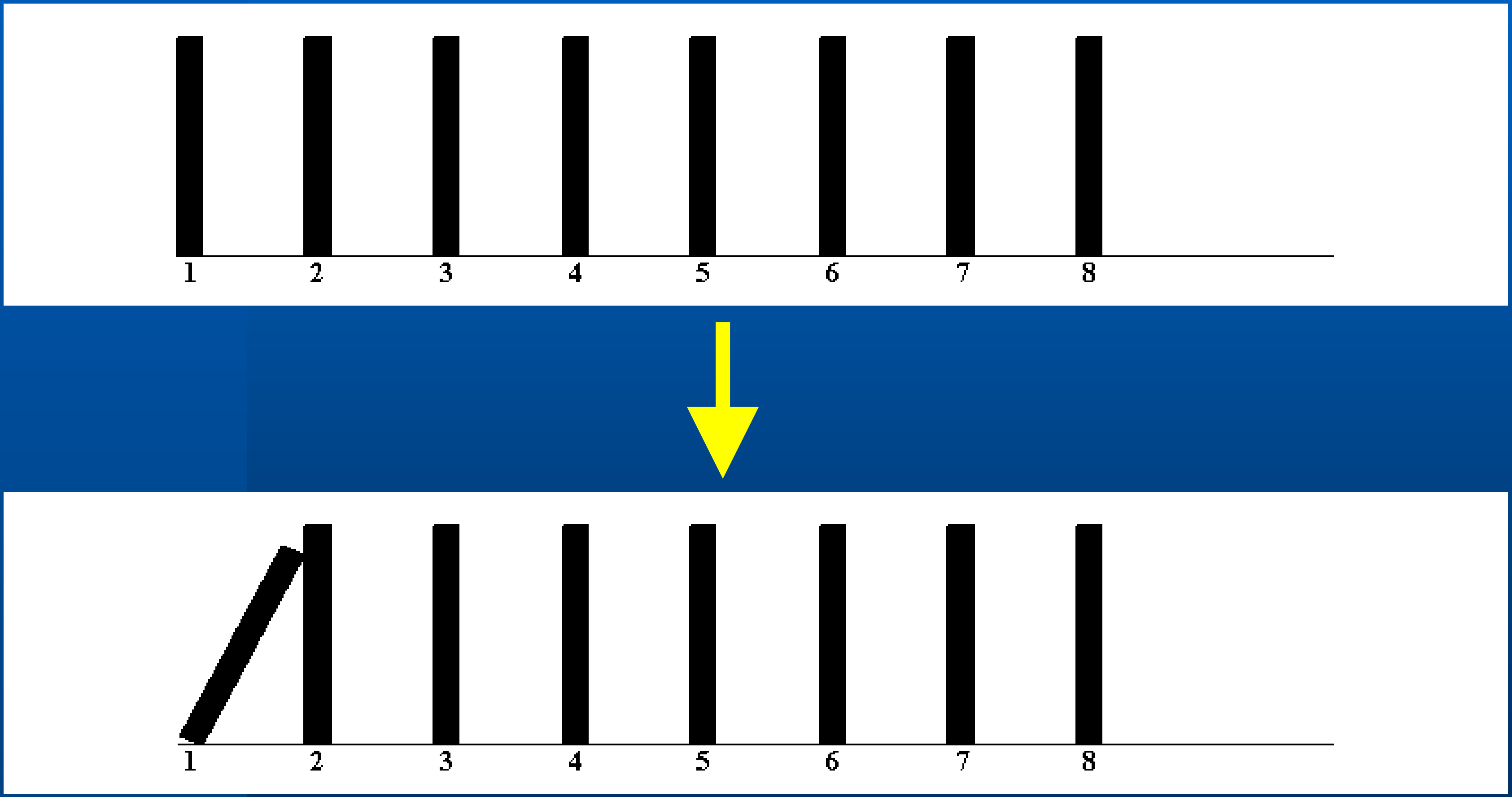
Discrete Mathematics for Computer Science

Lecture 7

Mathematical Induction



Example 1

Prove with mathematical induction that for all $n \in N$, $n \geq 1$:

$$\sum_{i=1}^{n}(2i-1)=n^2$$

Proof:

1. **Basis Step:**

The statement is true for $n = 1$, because:

$$\sum_{i=1}^{1}(2i-1) = (2\cdot 1-1) = 1$$ **and also** $$1^2 = 1$$

Proof (continued)

2. <u>**Induction Step**</u>:

Let $k \geq 1$ and suppose that:

$$\sum_{i=1}^{k}(2i-1)=k^2$$ **IH (Induction Hypothesis)**

We will (try to) show that IH implies:

$$\sum_{i=1}^{k+1}(2i-1)=(k+1)^2$$

Proof (continued)

2. Show that for any $k \ge 1$

$$\sum_{i=1}^{k}(2i-1)=k^2$$ **IH** implies $$\sum_{i=1}^{k+1}(2i-1)=(k+1)^2$$

Well: $$\sum_{i=1}^{k+1}(2i-1) = \sum_{i=1}^{k}(2i-1)+\left(2(k+1)-1\right)$$

IH

$$= k^2+\left(2(k+1)-1\right)$$

$$= k^2+2k+1$$

$$= (k+1)^2$$

From 1 and 2 we conclude that

for <u>all</u> $n \ge 1$: $$\sum_{i=1}^{n}(2i-1)=n^2$$

Example 2

$0^2 + 0 + 41 = 41$ → prime!

$1^2 + 1 + 41 = 43$ → prime!

$2^2 + 2 + 41 = 47$ → prime!

$3^2 + 3 + 41 = 53$ → prime!

$4^2 + 4 + 41 = 61$ → prime!

Statement:

For all $n \geq 0$: $n^2 + n + 41$ is prime

Proof (?)

For all *n* ≥ 0: $n^2 + n + 41$ is prime

1. Basis Step:

The statement is true for *n* = 0, because

$0^2 + 0 + 41 = 41$ is prime.

2. Induction Step

Let *k* ≥ 0, and suppose that

$k^2 + k + 41$ is prime **IH**

We will (try to) show that IH implies:

$(k+1)^2 + (k+1) + 41$ is prime

Proof (?) (continued)

2. Show that for any $k \geq 0$

$k^2 + k + 41$ is prime **IH**

implies $(k+1)^2 + (k+1) + 41$ is prime

Well: $(k+1)^2 + (k+1) + 41$ $= k^2 + 2k + 1 + k + 1 + 41$

$= \left(k^2 + k + 41\right) + 2k + 2$ (*)

The first part is prime by IH.

But does this imply that also (*) is prime ???

No! The statement is not true for all *n*.

E.g. if we take *n* = 41:

$41^2 + 41 + 41$ $= 43 \cdot 41$ is not a prime!

Mathematical Induction

Theorem 4.1

Let, for each $n \in Z^+$, $S(n)$ be an open statement.

(so the truth value of $S(n)$ depends on n)

Then $S(n)$ is true for all $n \in Z^+$ if:

1. $S(1)$ is true and
2. For all $k \in Z^+$ the following implication holds: $S(k) \to S(k+1)$

(1. is the Basis Step, 2. is the Induction Step)

Well-Orderings Principle of Z^+

Every nonempty subset of Z^+ contains a smallest element.

Remark:

- This is also true for N, but not for Q^+ and R^+.
- This principle is used in the **Principle of Mathematical Induction**

1. $S(1)$ is true and

2. For all $k \in Z^+$: $S(k) \rightarrow S(k+1)$

Proof:

Let $F = \{t \in Z^+ \mid S(t) \text{ is false}\}$ We will show that $F = \varnothing$.

We will show this by contradiction: Suppose: $F \neq \varnothing$.

Obviously $F \subseteq Z^+$.

So, by the Well Orderings Principle of Z^+, F has a smallest element. Let $j \in F$ be this smallest element.

Then: $j > 1$, since $S(1)$ is true.

Moreover, $S(j–1)$ is true, because j is the smallest value for which $S(j)$ is false.

But this contradicts 2. (take $k = j – 1$),
since $S(j–1)$ true implies $S(j)$ true. Therefore $F = \varnothing$.

Remarks to Theorem 4.1

1. The Principle of Mathematical Induction not only holds for Z^+ , but for each set of the form:

$$\{n_0, n_0+1, n_0+2, n_0+3, \cdots\cdots\}$$ with $n_0 \in Z$

2. We have seen that the Induction Step in a proof by Mathematical Induction is essential. But this is also the case for the Basis Step!

For example, by only verifying the Induction Step, one could "prove" the nonsense statement:

$$\sum_{i=1}^{n}(2i-1) = n^2 + 137$$ for all $n \in Z^+$

Example 3

Prove with mathematical induction that for all $n \in N$, $3 \mid (n^3 - n)$ **(i.e, 3 divides $n^3 - n$)**

In other words: there exists a $t \in Z$ with $n^3 - n = 3t$

Proof:

1. **Basis Step for $n = 0$:**

$$0^3 - 0 = 0 = 3 \cdot 0$$

(take $t = 0$)

Proof (continued)

2. **Induction Step**:

Let $k \geq 0$ and suppose there exists $t \in Z$ with

$$k^3 - k = 3t$$ **IH**

We will show that there exists an $s \in Z$ with:

$$(k+1)^3 - (k+1) = 3s$$

Well: $(k+1)^3 - (k+1)$

$$= k^3 + 3k^2 + 3k + 1 - (k+1)$$

IH

$$= \left(k^3 - k\right) + 3k^2 + 3k \quad = 3t + 3k^2 + 3k$$

$$= 3\left(t + k^2 + k\right)$$ **(so: take $s = t + k^2 + k$)**

Example 4

Prove with mathematical induction that for all $n \in N$, $n \geq 5$: $n^2 < 2^n$

Proof:

1. Basis Step:

The statement is true for $n = 5$, because:

$5^2 = 25$ **and** $2^5 = 32$

So $5^2 < 2^5$

Proof (continued)

2. **Induction Step**:

Let $k \ge 5$ and suppose that $k^2 < 2^k$ **IH**

We will show that: $(k+1)^2 < 2^{k+1}$

Well: $(k+1)^2 = k^2 + 2k + 1 \underset{\text{IH}}{<} 2^k + 2k + 1$

So it suffices to show that

$$2^k + 2k + 1 \le 2^{k+1} \left(= 2 \cdot 2^k \right)$$

So it suffices to show that $2k + 1 \le 2^k$

This is straightforward, since $k \ge 5$:

$$2k + 1 < 2k + k = 3k < 5k \le k \cdot k = k^2 \overset{\text{IH}}{<} 2^k$$

Example 5 (Harmonic numbers)

Let, for $n \geq 1$, $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ $= \sum_{i=1}^{n} \frac{1}{i}$

Prove with mathematical induction that for all $n \geq 0$, $H_{2^n} \leq n + 1$

Proof:

1. Basis Step:

The statement is true for $n = 0$, because:

$H_{2^0} = H_1 = \sum_{i=1}^{1} \frac{1}{i} = 1$ **and** $1 \leq 0 + 1$

Proof (continued)

$$H_n = \sum_{i=1}^{n} \frac{1}{i}$$

2. **Induction Step**:

Let *k* ≥ 0 and suppose that $H_{2^k} \leq k+1$ **IH**

We will show that IH implies $H_{2^{k+1}} \leq (k+1)+1$ **IH**

Well: $H_{2^{k+1}} = \sum_{i=1}^{2^{k+1}} \frac{1}{i}$ $= \sum_{i=1}^{2^k} \frac{1}{i} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i}$ $\leq k+1+ \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i}$

Now it remains to show that $\sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} \leq 1$

In expanded form: $\frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}} \leq 1$

This inequality holds, since its left hand side consists of 2^k terms which are all ≤ $1/2^k$:

$$\frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^k+2^k} \leq \frac{1}{2^k} + \cdots + \frac{1}{2^k} = \frac{2^k}{2^k} = 1$$