Discrete Mathematics for Computer Science

Lecture 6

Theorem 3.4

The following statements are equivalent for sets A and B:

a) $A \subseteq B$

b) $A \cup B = B$

c) $A \cap B = A$

d) $\overline{B} \subseteq \overline{A}$

So:

$$A \subseteq B \iff A \cup B = B \iff A \cap B = A \iff \overline{B} \subseteq \overline{A}$$

Structure of the proof

$$A \subseteq B \Leftrightarrow A \cup B = B \Leftrightarrow A \cap B = A \Leftrightarrow \overline{B} \subseteq \overline{A}$$

Note that is suffices to prove the implications:

(1) $A \subseteq B \Rightarrow A \cup B = B$ (2) $A \cup B = B \Rightarrow A \cap B = A$

(3) $A \cap B = A \Rightarrow \overline{B} \subseteq \overline{A}$ (4) $\overline{B} \subseteq \overline{A} \Rightarrow A \subseteq B$

(cf. book for the complete proof)

Proofs concerning Sets (1)

Often statements concerning sets are translated into statements in logic, using the following substitutions:

$A \rightarrow x \in A$

$B \rightarrow x \in B$

$A \cap B \rightarrow x \in A \wedge x \in B$

$A \cup B \rightarrow x \in A \vee x \in B$

$\overline{A} \rightarrow \neg(x \in A) \ (x \notin A)$

$A - B \rightarrow x \in A \wedge \neg(x \in B)$

$U \rightarrow x \in U \ (T_0)$

$\varnothing \rightarrow x \in \varnothing \ (F_0)$

$= \rightarrow \Leftrightarrow$

$\subseteq \rightarrow \Rightarrow$

$A \Delta B \rightarrow x \in (A \cup B) \wedge \neg x \in (A \cap B)$

Proofs concerning Sets (2)

Example

$$(A \cup B) - C \subseteq (A - C) \cup B$$

can be written as:

$$(x \in A \vee x \in B) \wedge \neg(x \in C) \Rightarrow (x \in A \wedge \neg(x \in C)) \vee x \in B$$

Substituting

p **for** ***x*∈*A*** **;** ***q*** **for** ***x*∈*B*** **and** ***r*** **for** ***x*∈*C*****, we get**

$$(p \vee q) \wedge \neg r \Rightarrow (p \wedge \neg r) \vee q$$

which can be proved using L-laws and R-rules.

Proofs concerning Sets (3)

A Venn-diagram is not considered as a proof !!!

But they can help you in examining statements concerning sets.

Laws of Set Theory

The substitutions on the previous slide transform the Laws of Logic into the Laws of Set Theory (see next slide).

1) $\neg\neg p \Leftrightarrow p$

2) $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

3) $p \vee q \Leftrightarrow q \vee p$
$p \wedge q \Leftrightarrow q \wedge p$

4) $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r^{\dagger}$
$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

5) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

6) $p \vee p \Leftrightarrow p$
$p \wedge p \Leftrightarrow p$

7) $p \vee F_0 \Leftrightarrow p$
$p \wedge T_0 \Leftrightarrow p$

8) $p \vee \neg p \Leftrightarrow T_0$
$p \wedge \neg p \Leftrightarrow F_0$

9) $p \vee T_0 \Leftrightarrow T_0$
$p \wedge F_0 \Leftrightarrow F_0$

10) $p \vee (p \wedge q) \Leftrightarrow p$
$p \wedge (p \vee q) \Leftrightarrow p$

1) $\overline{\overline{A}} = A$

2) $\overline{A \cup B} = \overline{A} \cap \overline{B}$
$\overline{A \cap B} = \overline{A} \cup \overline{B}$

3) $A \cup B = B \cup A$
$A \cap B = B \cap A$

4) $A \cup (B \cup C) = (A \cup B) \cup C$
$A \cap (B \cap C) = (A \cap B) \cap C$

5) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

6) $A \cup A = A$
$A \cap A = A$

7) $A \cup \emptyset = A$
$A \cap \mathscr{U} = A$

8) $A \cup \overline{A} = \mathscr{U}$
$A \cap \overline{A} = \emptyset$

9) $A \cup \mathscr{U} = \mathscr{U}$
$A \cap \emptyset = \emptyset$

10) $A \cup (A \cap B) = A$
$A \cap (A \cup B) = A$

Theorem 3.5 (Principle of Duality)

Consider an equality of two expressions concening sets, merely containing the operations

$\neg$, $\cup$ and $\cap$. Then the interchangement of

- Each $\cup$ by $\cap$
- Each $\cap$ by $\cup$
- Each U by $\varnothing$
- Each $\varnothing$ by U

results in a valid equality: the dual equality.

Examples:

Cf. pairs S2 – S10 of Laws of Set Theory.

Example of a proof of a Statement on Sets

Prove the following equivalence:

$$A \subseteq B \quad \Leftrightarrow \quad \overline{A} \cup B = U$$

$$A \subseteq B \Leftrightarrow \overline{A} \cup B = U$$

Solution

First we will show:

$$A \subseteq B \Rightarrow \overline{A} \cup B = U$$

And then: $\overline{A} \cup B = U \Rightarrow A \subseteq B$

(1) Suppose: $A \subseteq B$

We must show that: $\overline{A} \cup B = U$

So: $\overline{A} \cup B \subseteq U$ **and** $U \subseteq \overline{A} \cup B$

$$A \subseteq B \quad \Rightarrow \quad \overline{A} \cup B = U$$

Solution (continued)

We must show that: $\overline{A} \cup B \subseteq U$ **and** $U \subseteq \overline{A} \cup B$

By definition of universe, we have: $\overline{A} \cup B \subseteq U$

So it remains to show that: $U \subseteq \overline{A} \cup B$

Let $x \in U$

We distinguish the cases $x \in \overline{A}$ and $x \notin \overline{A}$

If $x \in \overline{A}$ then obviously $x \in \overline{A} \cup B$

If $x \notin \overline{A}$ then $x \in A$ and so $x \in B$ since $A \subseteq B$

So then again $x \in \overline{A} \cup B$

Hence, in any case $x \in \overline{A} \cup B$ So $U \subseteq \overline{A} \cup B$

$$\overline{A} \cup B = U \;\Rightarrow\; A \subseteq B$$

Solution (continued)

(2) Proof of: $\overline{A} \cup B = U \;\Rightarrow\; A \subseteq B$

Suppose: $\overline{A} \cup B = U$

We must show that: $A \subseteq B$

Let $x \in A$ **Then certainly** $x \in U$

And so $x \in \overline{A} \cup B$ **since** $\overline{A} \cup B = U$

So: $x \in \overline{A}$ **or** $x \in B$

Since $x \in A$ **we have:** $x \notin \overline{A}$

And so necessarily: $x \in B$

Union/Intersection of Sets

Let *U* be a universe and let *I* a nonempty (index)set.

Let, for each $i \in I$, $A_i \subseteq U$. Then

$$\bigcup_{i \in I} A_i = \{x \mid x \in A_i \text{ for at least one } i \in I\}$$

$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\}$$

Example

Let, for $n \in Z^{+}$: $= A_n = \left(-n, \frac{1}{n}\right]$

Determine: **Solution:**

a. $\bigcup_{n=1}^{7} A_n$ $\left(-7, 1\right]$

b. $\bigcap_{n=1}^{12} A_n$ $\left(-1, \frac{1}{12}\right]$

c. $\bigcup_{n=1}^{\infty} A_n$ $\left(-\infty, 1\right]$

d. $\bigcap_{n=1}^{\infty} A_n$ $\left(-1, 0\right]$

Generalisation “DeMorgan”(S2)

Theorem 3.6

a. $\overline{\bigcup_{i\in I} A_i} = \bigcap_{i\in I} \overline{A_i}$

b. $\overline{\bigcap_{i\in I} A_i} = \bigcup_{i\in I} \overline{A_i}$

Proof:

a. $x \in \overline{\bigcup_{i\in I} A_i} \Leftrightarrow x \notin \bigcup_{i\in I} A_i \Leftrightarrow \neg\exists i\,(x \in A_i)$

$\Leftrightarrow \forall i\,\neg(x \in A_i) \Leftrightarrow \forall i\,(x \in \overline{A_i}) \Leftrightarrow x \in \bigcap_{i\in I} \overline{A_i}$

b. Exercise 3.2.20