Discrete Mathematics for Computer Science

Lecture 5

Sets and elements

Consider a set *A* in a universe *U*.

$x \in A$ **means: *x* is an element of *A* (*x* is in *A*)**

$x \notin A$ **means: *x* is not an element of *A***

Standard Sets
(see page 133-134 for Z^+, Q^*, etc)

N: **natural numbers**

$$N = \{0,1,2,3,\cdots\}$$

Z: **integers**

$$Z = \{-3,-2,-1,0,1,2,3,\cdots\}$$

Q: **rational numbers**

$$Q = \left\{ \frac{a}{b} \,\middle|\, a,b \in Z \wedge b \neq 0 \right\}$$

R: **real numbers**

$$R = (-\infty, \infty)$$

C: **complex numbers**

$$C = \{x + iy \mid x, y \in R \wedge i^2 = -1\}$$

Intervals in *R*

$$[a,b] = \{x \in R \mid a \le x \le b\}$$

$$[a,b) = \{x \in R \mid a \le x < b\}$$

$$(a,b] = \{x \in R \mid a < x \le b\}$$

$$(a,b) = \{x \in R \mid a < x < b\}$$

Different notations (1)

$$\{2,3,4,5,6,7\}$$

$$\{x \in Z \mid 2 \le x \le 7\}$$

$$\{x \in N \mid 2 \le x \le 7\}$$

$$\{x \mid x \in Z \wedge 2 \le x \le 7\}$$

$$\{x \mid x \in Z, 2 \le x \le 7\}$$

Different notations (2)

Finite set:

$$\{0,1,4,9,16,\cdots,100\} = \{x^2 \mid x \in N, x \leq 10\}$$

Infinite set:

$$\{1,3,5,7,\cdots\} = \{2k+1 \mid k \in N\}$$

Cardinality

The number of elements of a finite set *A* is the <u>cardinality</u> of *A*, and is denoted by: |*A*|.

<u>Example</u>:

$$\left|\{0,1,4,9,16,\cdots,100\}\right| = 11$$

Subsets (1)

$A \subseteq B$ means: "*A* is a **subset** of *B*"

So: $\forall x \left[\left(x \in A \right) \rightarrow \left(x \in B \right) \right]$

$A \not\subseteq B$ means: "*A* is **not** a subset of *B*"

So: $\neg \forall x \left[\left(x \in A \right) \rightarrow \left(x \in B \right) \right]$

Subsets (2)

$A \not\subseteq B$

$A \not\subseteq B$ **means:** $\neg \forall x \left[(x \in A) \rightarrow (x \in B) \right]$

Simplifying this expression yields:

$\Leftrightarrow \quad \exists x \neg \left[(x \in A) \rightarrow (x \in B) \right]$ **N1**

$\Leftrightarrow \quad \exists x \neg \left[\neg (x \in A) \vee (x \in B) \right]$ **L12**

$\Leftrightarrow \quad \exists x \left[\neg\neg (x \in A) \wedge \neg (x \in B) \right]$ **L2**

$\Leftrightarrow \quad \exists x \left[(x \in A) \wedge \neg (x \in B) \right]$ **L1**

$\Leftrightarrow \quad \exists x \left[(x \in A) \wedge (x \notin B) \right]$

Subsets (3)

$A \subset B$ **means:**

"*A* is a strict (proper) subset of *B*"

So: $(A \subseteq B) \wedge (A \neq B)$

And $A \not\subset B$ **means:** $\neg(A \subset B)$

So: $\neg((A \subseteq B) \wedge (A \neq B))$

So, by L2: $(A \not\subseteq B) \vee (A = B)$

Examples

Note: a set can also be an element of another set!

$A = \{1,2,3\}$ $B = \{2,3,4\}$ $C = \{1,2,3,4,5\}$

$D = \{1,2,\{1,2,3\}\}$

Then: $A \not\subseteq B$ $B \not\subseteq A$ $|C| = 5$ $|D| = 3$

$A \subseteq C$ $A \subset C$ $A \notin C$

$A \in D$ $A \not\subseteq D$ $\{A\} \notin D$ $\{A\} \subseteq D$

Equality of Sets

$A = B$ **means:** $(A \subseteq B) \wedge (B \subseteq A)$

So:

$$\forall x[(x \in A) \rightarrow (x \in B)] \wedge \forall x[(x \in B) \rightarrow (x \in A)]$$

How to prove that two sets *A* and *B* are equal?

Show that <u>each</u> element of *A* is in *B* <u>and</u> that <u>each</u> element of *B* is in *A* !

Example

Let $A = \{5k - 1 \mid k \in Z\}$ and $B = \{5m + 4 \mid m \in Z\}$

Then: $A = B$.

Proof: (consists of 2 parts (i) and (ii))

(i) $A \subseteq B$ (each element of A is in B)

(ii) $B \subseteq A$ (each element of B is in A)

$A = \{5k - 1 \mid k \in Z\}$ **and** $B = \{5m + 4 \mid m \in Z\}$

Solution (i)

(i) Proof of $A \subseteq B$

Let $x \in A$**.**

Then there exists a $k \in Z$ **such that** $x = 5k - 1$

In order to show that $x \in B$ **we must show that there exists an** $m \in Z$ **such that** $x = 5m + 4$

$5m + 4 = 5k - 1$ **implies** $5m = 5k - 5$

So take: $m = k - 1$

$A = \{5k - 1 \mid k \in Z\}$ **and** $B = \{5m + 4 \mid m \in Z\}$

Solution (ii)

(ii) Proof of $B \subseteq A$

Let $x \in B$**.**

Then there exists an $m \in Z$ **such that** $x = 5m + 4$

In order to show that $x \in A$ **we must show that there exists a** $k \in Z$ **such that** $x = 5k - 1$

$5k - 1 = 5m + 4$ **implies** $5k = 5m + 5$

So take: $k = m + 1$

Inequality of Sets

$A \neq B$ **means:** $(A \not\subseteq B) \vee (B \not\subseteq A)$

How to prove that two sets *A* and *B* are <u>not</u> equal?

Find an element that is in *A* but not in *B* or
Find an element that is in *B* but not in *A* !

Example

Let $A = \{4k+1 \mid k \in Z\}$ and $B = \{2m-1 \mid m \in Z\}$

Determine whether: $A = B$ or $A \neq B$.

Solution:

We have $A \neq B$, because:

$3 \in B$ (take $m = 2$), but $3 \notin A$ since

$4k+1=3$ would imply $k = \frac{1}{2}$ but $\frac{1}{2} \notin Z$

Remark: verify (prove!) however that $A \subseteq B$

The empty set: $\varnothing$

The empty set is the unique set with cardinality 0.

So: $|\varnothing| = 0$

But: $\varnothing \neq \{0\}$

And: $\varnothing \neq \{\varnothing\}$

Theorem 3.2

For each set *A* we have: $\emptyset \subseteq A$

And also:

If $A \neq \emptyset$ then $\emptyset \subset A$

(proof: in the book)

Powerset (1)

The Powerset of a set A is the set of all subsets of A.

Notation: $P(A)$.

So: $P(A) = \{B \mid B \subseteq A\}$

Example: $A = \{1,2,3\}$

$$P(A) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

Powerset (2)

For each set A we have: $$|P(A)| = 2^{|A|}$$

Proof:

Each subset $B \subseteq A$ can be constructed as follows: Decide for each element $x \in A$ whether $x \in B$ or $x \notin B$. So 2 choices for each element. Since A has $|A|$ elements, there are exactly $2^{|A|}$ ways to construct a subset B.

Theorem 3.1

Let *A*, *B* and *C* be sets. Then:

If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

If $A \subset B$ and $B \subseteq C$ then $A \subset C$

If $A \subseteq B$ and $B \subset C$ then $A \subset C$

If $A \subset B$ and $B \subset C$ then $A \subset C$

(proof: in the book)

§3.2: Operations on Sets

Consider sets *A* and *B* in a universe *U*.

Union of *A* and *B*: $A \cup B = \{x \mid x \in A \vee x \in B\}$

Intersection of *A* and *B*: $A \cap B = \{x \mid x \in A \wedge x \in B\}$

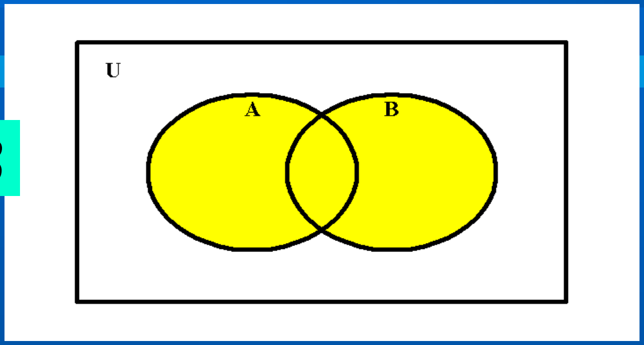
Difference of *A* and *B*: $A - B = \{x \mid x \in A \wedge x \notin B\}$

Symmetric Difference of *A* and *B*: $A \Delta B = \{x \mid x \in A \cup B \wedge x \notin A \cap B\}$

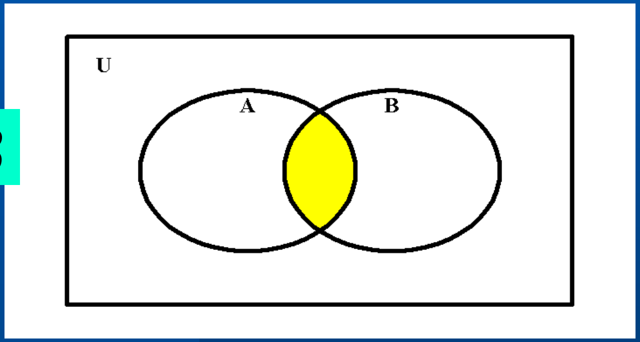
Complement of *A*: $\overline{A} = U - A = \{x \mid x \notin A\}$

Venn-diagrams:

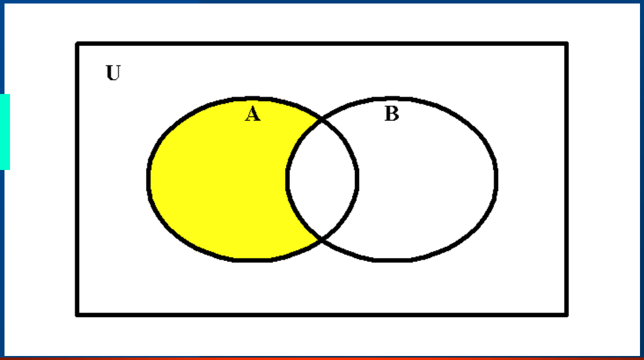
$A \cup B$



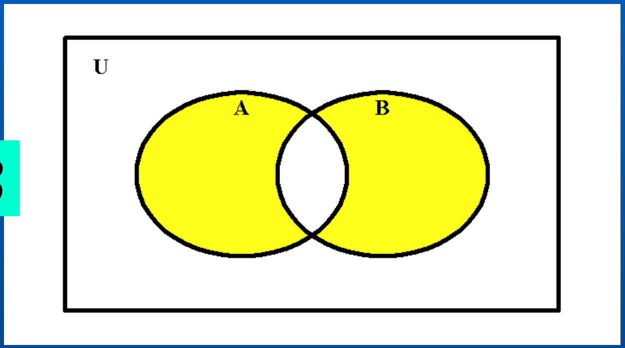
$A \cap B$



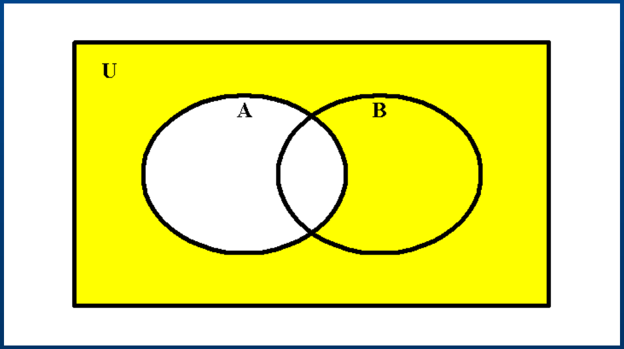
$A - B$



$A \Delta B$



$\overline{A}$



Disjoint Sets

Sets *A* and *B* are called <u>disjoint</u> if: $A \cap B = \varnothing$

<u>Theorem 3.3</u>: *A* and *B* are sets in a universe *U*.

Then: *A* and *B* are disjoint if and only if

$$A \cup B = A \Delta B$$

So: $A \cap B = \varnothing \quad \Leftrightarrow \quad A \cup B = A \Delta B$

$$A \cap B = \varnothing \quad \Leftrightarrow \quad A \cup B = A \Delta B$$

Proof:

First we show: $A \cap B = \varnothing \quad \Rightarrow \quad A \cup B = A \Delta B$

And then: $A \cup B = A \Delta B \quad \Rightarrow \quad A \cap B = \varnothing$ (L11)

Part 1: Suppose that $A \cap B = \varnothing$

We must show that: $A \cup B = A \Delta B$

So, we must show that:

$A \cup B \subseteq A \Delta B$ **and** $A \Delta B \subseteq A \cup B$

(a) Proof of $A \cup B \subseteq A \Delta B$

Let $x \in A \cup B$

Since $A \cap B = \varnothing$ we have $x \notin A \cap B$

So, by definition of Δ: $x \in A \Delta B$

$$A \cap B = \varnothing \quad \Leftrightarrow \quad A \cup B = A \Delta B$$

Proof of $A \cap B = \varnothing \quad \Rightarrow \quad A \cup B = A \Delta B$

(b) Proof of $A \Delta B \subseteq A \cup B$

Let $x \in A \Delta B$

Then $x \in A \cup B$ **and** $x \notin A \cap B$

So certainly: $x \in A \cup B$

(a) and (b) complete the proof of Part 1:

$$A \cap B = \varnothing \quad \Rightarrow \quad A \cup B = A \Delta B$$

$$A \cap B = \emptyset \quad \Leftrightarrow \quad A \cup B = A \Delta B$$

Proof of Part 2 $A \cup B = A \Delta B \quad \Rightarrow \quad A \cap B = \emptyset$

Suppose $A \cup B = A \Delta B$

We must show that: $A \cap B = \emptyset$

We will prove this by contradiction: suppose that $A \cap B \neq \emptyset$

Then there exists an $x \in U$ such that $x \in A \cap B$

Then: $x \in A$ and $x \in B$ So certainly $x \in A \cup B$

But $x \notin A \Delta B$ since $x \in A \cap B$

Therefore: $A \cup B \neq A \Delta B$

But this contradicts $A \cup B = A \Delta B$

Therefore, the assumption $A \cap B \neq \emptyset$ must be wrong.

So we must have: $A \cap B = \emptyset$

This completes the proof of Part 2.