Discrete Mathematics for Computer Science

Lecture 4

Interchanging quantifiers

In each universe *U*:

$$\forall x \forall y\, p(x, y) \quad \Leftrightarrow \quad \forall y \forall x\, p(x, y)$$

$$\exists x \exists y\, p(x, y) \quad \Leftrightarrow \quad \exists y \exists x\, p(x, y)$$

However, false is:

$$\forall x \exists y\, p(x, y) \quad \Leftrightarrow \quad \exists y \forall x\, p(x, y)$$

Counterexample: $U = (0, \infty)$ **and** $p(x,y)$**:** $x \cdot y = 1$

§2.5: Four additional rules of inference w.r.t. quantifiers (cf. formula sheet)

Rule of Universal Specification

U1 $$\frac{\forall x\, p(x)}{\therefore p(c)}$$ **for <u>any</u>** $c \in U$

Rule of Existential Specification

U2 $$\frac{\exists x\, p(x)}{\therefore\, p(c)}$$ **for <u>some</u>** $c \in U$

Rule of Universal Generalization

U3 $$\frac{p(c)}{\therefore \forall x\, p(x)}$$ **for <u>any</u>** $c \in U$

Rule of Existential Generalization

U4

$$\frac{p(c)}{\therefore \exists x\, p(x)}$$

for <u>some</u> $c \in U$

Example 1

Prove the validity of the following argument by using the "Laws of Logic", the "Rules of Inference" and the supplement w.r.t. quantifiers.

$$\exists x\big(p(x) \wedge q(x)\big) \quad \Rightarrow \quad \big(\exists x\, p(x) \wedge \exists x\, q(x)\big)$$

In tabular form:

$$\frac{\exists x\big(p(x) \wedge q(x)\big)}{\therefore \exists x\, p(x) \wedge \exists x\, q(x)}$$

$$\frac{\exists x\big(p(x) \wedge q(x)\big)}{\therefore \exists x\, p(x) \wedge \exists x\, q(x)}$$

Solution

(1)	$\exists x\big(p(x) \wedge q(x)\big)$	**Premise**
(2)	$p(c) \wedge q(c)$	**for <u>some</u> $c \in U$ (1), U2**
(3)	$p(c)$	**(2), R7**
(4)	$\exists x\, p(x)$	**(3), U4**
(5)	$q(c) \wedge p(c)$	**(2), L3**
(6)	$q(c)$	**(5), R7**
(7)	$\exists x\, q(x)$	**(6), U4**
(8)	$\therefore \exists x\, p(x) \wedge \exists x\, q(x)$	**(4),(7), R4**

Example 2

We just proved that in each universe *U*:

$$\exists x\big(p(x) \wedge q(x)\big) \quad \Rightarrow \quad \big(\exists x\, p(x) \wedge \exists x\, q(x)\big)$$

This implication cannot be reversed !!

Counterexample: $U = Z$;

$p(x)$: x is even and

$q(x)$: x is odd

Example 3

Prove the validity of the following argument by using the "Laws of Logic", the "Rules of Inference" and the supplement w.r.t. quantifiers.

$$\left(\exists x\, p(x) \vee \exists x\, q(x)\right) \quad \Leftrightarrow \quad \exists x\left(p(x) \vee q(x)\right)$$

$$\left(\exists x\, p(x) \vee \exists x\, q(x)\right) \Leftrightarrow \exists x\left(p(x) \vee q(x)\right)$$

Solution (1)

Use L11 to split the exercise into two parts:

$$\left(\exists x\, p(x) \vee \exists x\, q(x)\right) \Rightarrow \exists x\left(p(x) \vee q(x)\right)$$

and

$$\exists x\left(p(x) \vee q(x)\right) \Rightarrow \left(\exists x\, p(x) \vee \exists x\, q(x)\right)$$

In tabular form:

$$\frac{\exists x\, p(x) \vee \exists x\, q(x)}{\therefore \exists x\left(p(x) \vee q(x)\right)} \quad \textbf{and} \quad \frac{\exists x\left(p(x) \vee q(x)\right)}{\therefore \exists x\, p(x) \vee \exists x\, q(x)}$$

$$\frac{\exists x\big(p(x) \vee q(x)\big)}{\therefore \exists x\, p(x) \vee \exists x\, q(x)}$$

Solution (2)

(1)	$\exists x\big(p(x) \vee q(x)\big)$	**Premise**
(2)	$p(c) \vee q(c)$ for some $c \in U$	**(1), U2**
(3)	$(p(c)$ for some $c \in U) \rightarrow \exists x\, p(x)$	**U4**
(4)	$(q(c)$ for some $c \in U) \rightarrow \exists x\, q(x)$	**U4**
(5)	$\exists x\, p(x) \vee \exists x\, q(x)$	**(3),(4),(2), R11**

$$\frac{\exists x\, p(x) \vee \exists x\, q(x)}{\therefore \exists x \big(p(x) \vee q(x)\big)}$$

Solution (3)

(1) $\exists x\, p(x) \vee \exists x\, q(x)$ **Premise**

(2) $\exists x\, p(x) \rightarrow (p(c) \text{ for some } c \in U)$ **U2**

(3) $\exists x\, q(x) \rightarrow (q(d) \text{ for some } d \in U)$ **U2**

(4) $(p(c) \text{ for some } c \in U) \vee (q(d) \text{ for some } d \in U)$

(2),(3),(1), R11

$$\frac{\exists x\, p(x) \vee \exists x\, q(x)}{\therefore \exists x\big(p(x) \vee q(x)\big)}$$

Solution (4)

(4) $(p(c) \text{ for some } c \in U) \vee (q(d) \text{ for some } d \in U)$

(5) $(p(c) \text{ for some } c \in U) \rightarrow (p(c) \vee q(c) \text{ for some } c \in U)$ **R8**

(6) $(q(d) \text{ for some } d \in U) \rightarrow (q(d) \vee p(d) \text{ for some } d \in U)$ **R8**

(7) $(p(c) \vee q(c) \text{ for some } c \in U) \vee (q(d) \vee p(d) \text{ for some } d \in U)$

(5),(6),(4), R11

$$\frac{\exists x\, p(x) \vee \exists x\, q(x)}{\therefore \exists x\big(p(x) \vee q(x)\big)}$$

Solution (5)

(7) $(p(c) \vee q(c) \text{ for some } c \in U) \vee (q(d) \vee p(d) \text{ for some } d \in U)$

(8) $(p(c) \vee q(c) \text{ for some } c \in U) \vee (p(d) \vee q(d) \text{ for some } d \in U)$

(7), L3

(9) $(p(c) \vee q(c) \text{ for some } c \in U) \rightarrow \exists x \left(p(x) \vee q(x)\right)$ **U4**

(10) $(p(d) \vee q(d) \text{ for some } d \in U) \rightarrow \exists x \left(p(x) \vee q(x)\right)$ **U4**

(11) $\exists x(p(x) \vee q(x)) \vee \exists x(p(x) \vee q(x))$ **(9),(10),(8), R11**

(12) $\therefore \exists x(p(x) \vee q(x))$ **(11), L6**

Example 4

Prove the validity of the following argument by using the "Laws of Logic", the "Rules of Inference" and the supplement w.r.t. quantifiers.

$$\begin{array}{c} \forall x\big(\big(q(x) \vee r(x)\big) \rightarrow \neg p(x)\big) \\ \exists x\, p(x) \\ \hline \therefore \neg \forall x\, r(x) \end{array}$$

$$\forall x\big((q(x) \vee r(x)) \rightarrow \neg p(x)\big)$$
$$\exists x\, p(x)$$
$$\therefore \neg \forall x\, r(x)$$

Solution

(1) $\exists x\, p(x)$ **Premise**

(2) $p(c)$ **for <u>some</u> *c*** **(1), U2**

(3) $\forall x\big((q(x) \vee r(x)) \rightarrow \neg p(x)\big)$ **Premise**

(4) $(q(c) \vee r(c)) \rightarrow \neg p(c)$ **(3), U1 (same *c* as in (2))**

(5) $\neg\neg p(c)$ **(2), L1**

(6) $\neg(q(c) \vee r(c))$ **(4), (5), R3**

(7) $\neg q(c) \wedge \neg r(c)$ **(6), L2**

(8) $\neg r(c) \wedge \neg q(c)$ **(7), L3**

(9) $\neg r(c)$ **(8), R7**

(10) $\exists x \neg r(x)$ **(9), U4**

(11) $\neg \forall x\, r(x)$ **(10), N1**

Example 5

Show that the following argument is invalid.

$$\begin{array}{c} \forall x\, q(x) \\ \exists x\left(p(x) \wedge q(x)\right) \\ \hline \therefore \forall x\, p(x) \end{array}$$

$$\frac{\begin{array}{c}\forall x\, q(x) \\ \exists x\big(p(x) \wedge q(x)\big)\end{array}}{\therefore \forall x\, p(x)}$$

Solution

E.g, take: $U = N$

and

$p(x)$: x is even

$q(x)$: $x \in N$

Then both premises are true,

but the conclusion is false!

Example 6

Show that the following argument is invalid.

$$\begin{array}{c} \neg\exists x\left(p(x)\wedge\neg q(x)\right) \\ \exists x\left(q(x)\wedge r(x)\right) \\ \hline \therefore \exists x\left(p(x)\wedge r(x)\right) \end{array}$$

$$\frac{\begin{array}{c}\neg\exists x\big(p(x)\wedge\neg q(x)\big)\\ \exists x\big(q(x)\wedge r(x)\big)\end{array}}{\therefore \exists x\big(p(x)\wedge r(x)\big)}$$

Solution

E.g, take: $U = N$

and

$p(x)$: $x = 1$

$q(x)$: x is odd

$r(x)$: $x = 3$

Then both premises are true, but the conclusion is false!

Hint for exercise 2.5.7b

Split the exercise

$$\forall x\big(p(x) \wedge q(x)\big) \quad \Leftrightarrow \quad \big(\forall x\, p(x) \wedge \forall x\, q(x)\big)$$

into

$$\forall x\big(p(x) \wedge q(x)\big) \quad \Rightarrow \quad \big(\forall x\, p(x) \wedge \forall x\, q(x)\big)$$

and

$$\big(\forall x\, p(x) \wedge \forall x\, q(x)\big) \quad \Rightarrow \quad \forall x\big(p(x) \wedge q(x)\big)$$

Cf. Example 3, slide 10.