

# Discrete Mathematics for Computer Science

## Lecture 3

## Additional example §2.3

Prove the validity of the following argument using the “Laws of Logic” and the “Rules of Inference”.

$$\begin{array}{l} (\neg p \vee q) \rightarrow r \\ r \rightarrow (s \vee t) \\ \neg s \wedge \neg u \\ \neg u \rightarrow \neg t \\ \hline \therefore p \end{array}$$

$$\begin{array}{l}
 (\neg p \vee q) \rightarrow r \\
 r \rightarrow (s \vee t) \\
 \neg s \wedge \neg u \\
 \neg u \rightarrow \neg t \\
 \hline
 \therefore p
 \end{array}$$

# Solution

- (1)  $\neg s \wedge \neg u$  Premise
- (2)  $\neg u \wedge \neg s$  (1), L3
- (3)  $\neg u$  (2), R7
- (4)  $\neg u \rightarrow \neg t$  Premise
- (5)  $\neg t$  (4), (3), R1
- (6)  $\neg s$  (1), R7
- (7)  $\neg s \wedge \neg t$  (6), (5), R4
- (8)  $\neg(s \vee t)$  (7), L2
- (9)  $r \rightarrow (s \vee t)$  Premise

- (10)  $\neg r$  (9), (8), R3
- (11)  $(\neg p \vee q) \rightarrow r$  Premise
- (12)  $\neg(\neg p \vee q)$  (11), (10), R3
- (13)  $\neg\neg p \wedge \neg q$  (12), L2
- (14)  $p \wedge \neg q$  (13), L1
- (15)  $\therefore p$  (14), R7

## §2.4: Predicate logic

(Predicate = Property)

An Open statement is an assertion that contains one or more free variables, which can be chosen from a fixed universe  $U$ .

A statement without free variables is a closed statement (cf. §2.1 - §2.3)

The truth value of an open statement depends on the values of its variable(s).

# Examples

$$U = Z = \{0, 1, -1, 2, -2, \dots\}$$

- $p(x)$ :  $x$  is even
- $q(x)$ :  $x + 5 = 9$
- $r(x,y)$ :  $x + y$  is divisible by 7

$p(4)$  is true;  $p(7)$  is false;

$q(4)$  is true;  $q(7)$  is false

$q(1/2)$  is not defined ( $U = Z$ )

$r(5,9)$  is true;  $r(5,8)$  is false.

# The quantifiers $\forall$ and $\exists$

$\forall x p(x)$  means:

$p(x)$  is true for all  $x \in U$ .

$\exists x p(x)$  means:

there exists (at least) an  $x \in U$  for which  $p(x)$  is true.

“ $\forall$ ” is the universal quantifier

“ $\exists$ ” is the existential quantifier

# Truth values quantifiers

$\forall x p(x)$	
True	if $p(a)$ is true for <u>each</u> $a \in U$
False	if $p(a)$ false at least one $a \in U$

$\exists x p(x)$	
True	if $p(a)$ is true for at least one $a \in U$
False	if $p(a)$ false for <u>each</u> $a \in U$

# Examples

$$U = Z = \{0, 1, -1, 2, -2, \dots\}$$

- $p(x)$ :  $x$  is even
- $q(x)$ :  $x + 5 = 9$
- $r(x,y)$ :  $x + y$  is divisible by 7

$\forall x p(x)$  is false;  $\exists x p(x)$  is true;

$\forall x q(x)$  is false;  $\exists x q(x)$  is true;

$\forall x \forall y r(x,y)$  is false;  $\exists x \exists y r(x,y)$  is true;

$\exists x \forall y r(x,y)$  is false;

$\forall x \exists y r(x,y)$  is true.

**Note that the truth values depend on  $U$  !**

# Examples (continued)

$$U = Z = \{0, 1, -1, 2, -2, \dots\}$$

- $r(x,y): x + y$  is divisible by 7

So:

$\forall x \exists y$   $r(x,y)$  is true

And also:

$\forall x \exists y$   $\neg r(x,y)$  is true!

# Examples (continued)

$$U = Z = \{0, 1, -1, 2, -2, \dots\}$$

- $p(x): x > 0$
- $q(x): x$  is even
- $r(x): x$  is divisible by 3

Write the following statement in symbolic form and determine if it is true or false. If it is false, provide a counterexample.

1. There is no even number that is divisible by 3.

$$\neg(\exists x(q(x) \wedge r(x)))$$

False, take  $x = 6$ .

# Examples (continued)

$$U = Z = \{0, 1, -1, 2, -2, \dots\}$$

- $p(x): x > 0$
- $q(x): x$  is even
- $r(x): x$  is divisible by 3

Write the following statement in symbolic form and determine if it is true or false. If it is false, provide a counterexample.

2. Each positive even number is divisible by 3.

$$\forall x ((p(x) \wedge q(x)) \rightarrow r(x))$$

False, take  $x = 2$ .

# Examples

$A$  is an  $m \times n$ -matrix with entries  $a_{ij} \in R$  ( $1 \leq i \leq m$  ;  $1 \leq j \leq n$ ).

Write the following statement in symbolic form

1. Each column contains a positive entry.

$$\forall j \exists i (a_{ij} > 0)$$

2. There is no row in which the entries form a strictly decreasing sequence.

$$\neg \exists i \forall j \in \{1, \dots, n-1\} (a_{ij} > a_{i, j+1})$$

or

$$\neg \exists i \forall j (j \in \{1, \dots, n-1\} \rightarrow (a_{ij} > a_{i, j+1}))$$

# Laws concerning quantifiers

(cf. formula sheet)

$$\text{N1} \quad \neg(\forall x p(x)) \iff \exists x \neg p(x)$$

$$\text{N2} \quad \neg(\exists x p(x)) \iff \forall x \neg p(x)$$

# Example

Prove that:

$$\neg \exists x \forall y p(x, y) \iff \forall x \exists y \neg p(x, y)$$

$$\neg \exists x \forall y p(x, y) \quad \Leftrightarrow \quad \forall x \exists y \neg p(x, y)$$

## Solution

$$\neg \exists x \forall y p(x, y)$$

$$\Leftrightarrow \neg \exists x (\forall y p(x, y))$$

$$\Leftrightarrow \forall x (\neg \forall y p(x, y)) \quad \mathbf{N2}$$

$$\Leftrightarrow \forall x (\exists y \neg p(x, y)) \quad \mathbf{N1}$$

$$\Leftrightarrow \forall x \exists y \neg p(x, y)$$

# Example

Write the following statement about a  $m \times n$ -matrix  $A$  ( $m, n \geq 2$ ) in symbolic form. Then simplify the negation of the statement and write it in natural language. If the sum of two consecutive entries in a row equals 10, then the second column has an entry equal to 7.

**Solution:**

$$\forall i \forall j \in \{1, \dots, n - 1\} \\ \left( a_{ij} + a_{i,j+1} = 10 \rightarrow \exists k (a_{k2} = 7) \right)$$

# Solution (continued)

$$\forall i \forall j \in \{1, \dots, n - 1\}$$
$$\left( a_{ij} + a_{ij+1} = 10 \rightarrow \exists k (a_{k2} = 7) \right)$$

**Negation:**

$$\neg(\forall i \forall j \in \{1, \dots, n - 1\}$$
$$(a_{ij} + a_{ij+1} = 10 \rightarrow \exists k (a_{k2} = 7)))$$

$$\Leftrightarrow \exists i \exists j \in \{1, \dots, n - 1\}$$
$$\neg(a_{ij} + a_{ij+1} = 10 \rightarrow \exists k (a_{k2} = 7))$$

**N1 (2x)**

$$\Leftrightarrow \exists i \exists j \in \{1, \dots, n - 1\}$$
$$\neg(\neg(a_{ij} + a_{ij+1} = 10) \vee \exists k (a_{k2} = 7))$$

**L12**  
17

# Solution (continued)

$$\exists i \exists j \in \{1, \dots, n - 1\} \\ \neg(\neg(a_{ij} + a_{ij+1} = 10) \vee \exists k(a_{k2} = 7))$$

$$\Leftrightarrow \exists i \exists j \in \{1, \dots, n - 1\} \\ ((a_{ij} + a_{ij+1} = 10) \wedge \neg \exists k(a_{k2} = 7))$$

**L2, L1**

$$\Leftrightarrow \exists i \exists j \in \{1, \dots, n - 1\} \\ ((a_{ij} + a_{ij+1} = 10) \wedge \forall k \neg(a_{k2} = 7))$$

**N2**

$$\Leftrightarrow \exists i \exists j \in \{1, \dots, n - 1\} \\ \left( (a_{ij} + a_{ij+1} = 10) \wedge \forall k(a_{k2} \neq 7) \right)$$

## Solution (continued)

$$\exists i \exists j \in \{1, \dots, n - 1\} \\ \left( (a_{ij} + a_{ij+1} = 10) \wedge \forall k (a_{k2} \neq 7) \right)$$

**There exists a row in which the sum of two consecutive entries equals 10, but all entries in the second row are unequal to 7.**