Discrete Mathematics for Computer Science

Lecturer: Rico van Lingen

Lecture 1

Organisation (1)

- Size: 3 EC (± 84 hrs)
- ± 13 Colstructions (COL)
- Book: “Discrete and Combinatorial Mathematics and its Applications” (Ralph P. Grimaldi; 5th edition,

  ISBN: 0-321-21103-0 or 0-201-72634-3)

  This book will also be used in Module 7!

Organisation (2)

- Extension of Introduction to Mathematics (formerly known as Mathematics A)
  The reader of Intro to Math can be used as reference!
- Exercises in the course schedule that are not made in the lectures are considered as homework.

Organisation (3): Exam

- 2 partial tests that will be taken at the same time
- 1 resit in which you can do either one partial test or both
- All tests are in English.
- Electronic devices are not allowed.
- Formula-sheets will be handed out at the exam.

Program

Part 1:

Chapter 2: Logic

Chapter 3: Sets

Part 2:

Chapter 4: Mathematical Induction

Chapter 5: Relations and Functions

Chapter 7: Relations (continued)

Chapters 2, 3 and 4 are extensions of Intro to Math

What is Logic

Studies correctness of reasoning.

Example:

Socrates is a man

All men are mortal

Therefore: Socrates is Mortal

§2.1: Basic connectives and truth tables

A **proposition** is a statement that is unambiguously true or false (not both)

Examples:

- Tirana is the capital of Albania (proposition, true)
- $2 + 2 = 5$ (proposition, false)
- $x - 1 > 0$ (not a proposition)
- Do your best! (not a proposition)

Propositions are denoted with lower-case letters (*p*, *q*, *r*, etc)

Logical symbols

Connectives:

- $\neg p$ (negation) "not *p*"
- $p \wedge q$ (conjunction) "*p* and *q*"
- $p \vee q$ (disjunction) "*p* or *q* (or both)"
- $p \veebar q$ (exclusive disjunction) "either *p* or *q*"
- $p \rightarrow q$ (implication) "if *p* then *q*"
- $p \leftrightarrow q$ (equivalence) "*p* if and only if *q*"
- **Brackets:** (,) , [,] , etc

Truth tables for connectives

0 = false

1 = true

p	$\neg p$
0	1
1	0

Truth tables for connectives

$\neg$ not
$\wedge$ and
$\vee$ or
$\veebar$ either…or
$\rightarrow$ if …then
$\leftrightarrow$ if and only if

p	q	$p \wedge q$	$p \vee q$	$p \veebar q$	$p \rightarrow q$	$p \leftrightarrow q$
0	0	0	0	0	1	1
0	1	0	1	1	1	0
1	0	0	1	1	0	0
1	1	1	1	0	1	1

Semantics

Using truth tables one can verify the truthness of compount statements

$$\neg(p \wedge q) \rightarrow (p \vee \neg q)$$

$$\neg(p \wedge q) \rightarrow (p \vee \neg q)$$

Solution

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg q$	$p \vee \neg q$	$\neg(p \wedge q) \rightarrow (p \vee \neg q)$
0	0	0	1	1	1	1
0	1	0	1	0	0	0
1	0	0	1	1	1	1
1	1	1	0	0	1	1

Example 2:
Truth table for $\neg q \vee (\neg(r \rightarrow p) \leftrightarrow q)$

p	q	r	$\neg q$	$r \rightarrow p$	$\neg(r \rightarrow p)$	$\neg(r \rightarrow p) \leftrightarrow q$	$\neg q \vee (\neg(r \rightarrow p) \leftrightarrow q)$
0	0	0	1	1	0	1	1
0	0	1	1	0	1	0	1
0	1	0	0	1	0	0	0
0	1	1	0	0	1	1	1
1	0	0	1	1	0	1	1
1	0	1	1	1	0	1	1
1	1	0	0	1	0	0	0
1	1	1	0	1	0	0	0

Definitions

A **Tautology** is a statement for with the truth table consists merely of ones.

A **Contradiction** is a statement for with the truth table consists merely of zeros.

Tautology

$$(p \wedge q) \rightarrow p$$

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	1

Contradiction

$$(p \underline{\vee} q) \wedge (p \leftrightarrow q)$$

p	q	$p \underline{\vee} q$	$p \leftrightarrow q$	$(p \underline{\vee} q) \wedge (p \leftrightarrow q)$
0	0	0	1	0
0	1	1	0	0
1	0	1	0	0
1	1	0	1	0

§2.2: Logical equivalence

Notation of statements on slides:

To distinguish primitive statements from compound statements, we use the letters *p*, *q*, *r*, etc, for primitive and Greek letters (α, β, etc) for compound statements.

Logical equivalence

Two statements α and β are <u>logically equivalent</u> (notation: $\alpha \Leftrightarrow \beta$) if their truth tables are identical.

In other words: Two statements α and β are logically equivalent if the statement $\alpha \leftrightarrow \beta$ is a tautology.

Example

$$\neg(p \to q) \Leftrightarrow p \wedge \neg q$$

p	q	$p \to q$	$\neg(p \to q)$	$\neg q$	$p \wedge \neg q$
0	0	1	0	1	0
0	1	1	0	0	0
1	0	0	1	1	1
1	1	1	0	0	0

Other examples

Check using truth tables:

$$p \rightarrow q \quad \Leftrightarrow \quad \neg p \vee q$$

$$p \rightarrow q \quad \Leftrightarrow \quad \neg q \rightarrow \neg p$$

$$\neg(p \wedge q) \quad \Leftrightarrow \quad \neg p \vee \neg q$$

$$p \leftrightarrow q \quad \Leftrightarrow \quad (p \rightarrow q) \wedge (q \rightarrow p)$$

Tautologies and Contradictions

Note that all tautologies are logically equivalent. A tautology is denoted by T_0

Also all contradictions are logically equivalent. A contradiction is denoted by F_0

Formula sheets “Laws of Logic”

Contains a list of logically equivalent statements: L1 up to L13.

They form an extension of those on page 58-59 in the book.

You may use these laws at the exam (without first proving them using truth tables), just refer to them with L1, L2, … etc.

Dual statement

Consider a statement α that merely contains the connectives $\neg$, $\wedge$ and $\vee$.

The dual statement α^d of α is the statement that arises from α by replacing each:

- $\vee$ by $\wedge$
- $\wedge$ by $\vee$
- T_0 by F_0
- F_0 by T_0

Dual statement

Example:

If α**:** $\left(q \wedge \left(p \vee T_0\right)\right) \wedge \left(\neg q \vee F_0\right)$

then α^{d}**:** $\left(q \vee \left(p \wedge F_0\right)\right) \vee \left(\neg q \wedge T_0\right)$

Principle of Duality

Given two statements α and β that merely contain the connectives $\neg$, $\wedge$ and $\vee$.

Then we have:

if $\alpha \Leftrightarrow \beta$ then $\alpha^d \Leftrightarrow \beta^d$

Examples:

See the pairs L2 up to L10 of the "Laws of Logic" of the formula sheet.

Substitution rule 1

If we replace in a tautology α each occurrence of a primitive propositionletter p by the same statement β, then the resulting statement α^* is also a tautology.

Example:

$(p \wedge (p \to q)) \to q$ is a tautology (check yourself !),

So also $((q \to \neg s) \wedge ((q \to \neg s) \to q)) \to q$

is a tautology.

(replace each p by $q \to \neg s$).

Substitution rule 1

If we replace in a tautology α each occurrence of a primitive propositionletter p by the same statement β, then the resulting statement α^* is also a tautology.

Remarks:

1. This rule only applies to tautologies!

   E.g, $p \vee q$ is not equivalent to $(p \wedge q) \vee q$

2. Each occurence of p must be replaced!

   E.g, $p \vee \neg p$ is not equivalent to $(p \wedge q) \vee \neg p$

3. p must be primitive!

   E.g, $(r \vee s) \vee \neg s$ is not equivalent to $r \vee \neg s$

Substitution rule 2

Let α, β and γ be compound statements and suppose $\gamma \Leftrightarrow \beta$.

If we replace one or more occurrences of β in α by γ, then the resulting statement α^* satisfies: $\alpha \Leftrightarrow \alpha^*$.

Example: $\alpha : ((p \to q) \wedge (r \to (p \to q)))$

$\beta : (p \to q)$ $\gamma : (\neg p \vee q)$

By L12, we have: $(p \to q) \Leftrightarrow (\neg p \vee q)$

So: $((p \to q) \wedge (r \to (p \to q))) \Leftrightarrow ((p \to q) \wedge (r \to (\neg p \vee q)))$

Example "Laws of Logic"

Use the "Laws of Logic" to prove that:

$$\neg\left(p \rightarrow \left(\neg q \vee r\right)\right) \quad \Leftrightarrow \quad p \wedge q \wedge \neg r$$

Apply not more than one 1 law per step!

$$\neg(p \to (\neg q \lor r)) \quad \Leftrightarrow \quad p \land q \land \neg r$$

Solution (introduction)

1. Apply substitution rule 1 to the tautology L12

   $s \to t \Leftrightarrow \neg s \lor t$ with $s : p$ and $t : \neg q \lor r$

   This yields: $p \to (\neg q \lor r) \Leftrightarrow \neg p \lor (\neg q \lor r)$

2. Now apply substitution rule 2 to $\neg(p \to (\neg q \lor r))$

   and replace $p \to (\neg q \lor r)$

   by the equivalent statement $\neg p \lor (\neg q \lor r)$ (cf. Step 1)

   This yields: $\neg(p \to (\neg q \lor r)) \Leftrightarrow \neg(\neg p \lor (\neg q \lor r))$

We will abbreviate these arguments by:

$$\neg(p \to (\neg q \lor r)) \quad \Leftrightarrow \neg(\neg p \lor (\neg q \lor r)) \quad \text{L12}$$

$$\neg(p \to (\neg q \vee r)) \quad \Leftrightarrow \quad p \wedge q \wedge \neg r$$

Solution

Indicate in each step the Law that is applied (cf. formula sheet).

Don't forget the equivalence symbols!

No brackets means (by definition) that the connectives are applied from left to right!

$$\neg(p \to (\neg q \vee r))$$

$$\Leftrightarrow \neg(\neg p \vee (\neg q \vee r)) \qquad \text{L12}$$

$$\Leftrightarrow \neg\neg p \wedge \neg(\neg q \vee r) \qquad \text{L2}$$

$$\Leftrightarrow p \wedge \neg(\neg q \vee r) \qquad \text{L1}$$

$$\Leftrightarrow p \wedge (\neg\neg q \wedge \neg r) \qquad \text{L2}$$

$$\Leftrightarrow p \wedge (q \wedge \neg r) \qquad \text{L1}$$

$$\Leftrightarrow (p \wedge q) \wedge \neg r \qquad \text{L4}$$

$$\Leftrightarrow p \wedge q \wedge \neg r$$

Example "Laws of Logic"

Simplify the following statement using the "Laws of Logic".

$$\neg\left(\left(\left(p \vee q\right) \wedge r\right) \rightarrow \neg q\right)$$

Solution

Indicate in each step the Law that is applied (cf. formula sheet).

Don't forget the equivalence symbols!

$$\neg(((p \vee q) \wedge r) \rightarrow \neg q)$$

$$\Leftrightarrow \neg(\neg((p \vee q) \wedge r) \vee \neg q) \quad \textbf{L12}$$

$$\Leftrightarrow \neg\neg((p \vee q) \wedge r) \wedge \neg\neg q \quad \textbf{L2}$$

$$\Leftrightarrow ((p \vee q) \wedge r) \wedge q \quad \textbf{L1 (2x)}$$

$$\Leftrightarrow (p \vee q) \wedge (r \wedge q) \quad \textbf{L4}$$

$$\Leftrightarrow (p \vee q) \wedge (q \wedge r) \quad \textbf{L3}$$

$$\Leftrightarrow ((p \vee q) \wedge q) \wedge r \quad \textbf{L4}$$

Solution (continued)

$$\Leftrightarrow ((p \vee q) \wedge q) \wedge r$$

$$\Leftrightarrow (q \wedge (p \vee q)) \wedge r$$ **L3**

$$\Leftrightarrow (q \wedge (q \vee p)) \wedge r$$ **L3**

$$\Leftrightarrow q \wedge r$$ **L10**

Conclusion:

$$\neg(((p \vee q) \wedge r) \rightarrow \neg q) \quad \Leftrightarrow \quad q \wedge r$$

Application:
Simplifying switching networks

A switching network consists of terminals (*T*), wires and switches (*p*, *q*, *r*,…).

Example:

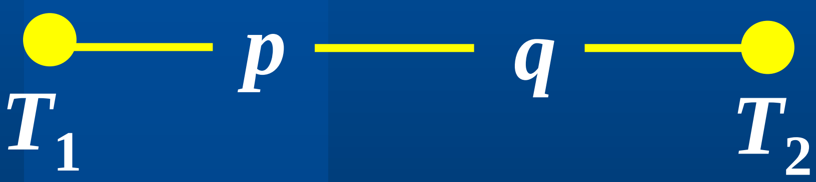
If $p = 0$, the switch is open and no current flows through it.

If $p = 1$, the switch is closed and current flows through it.

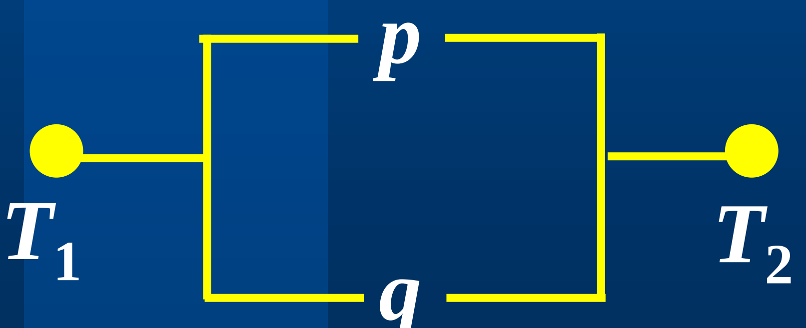
Simplifying switching networks

Examples:

One switch p



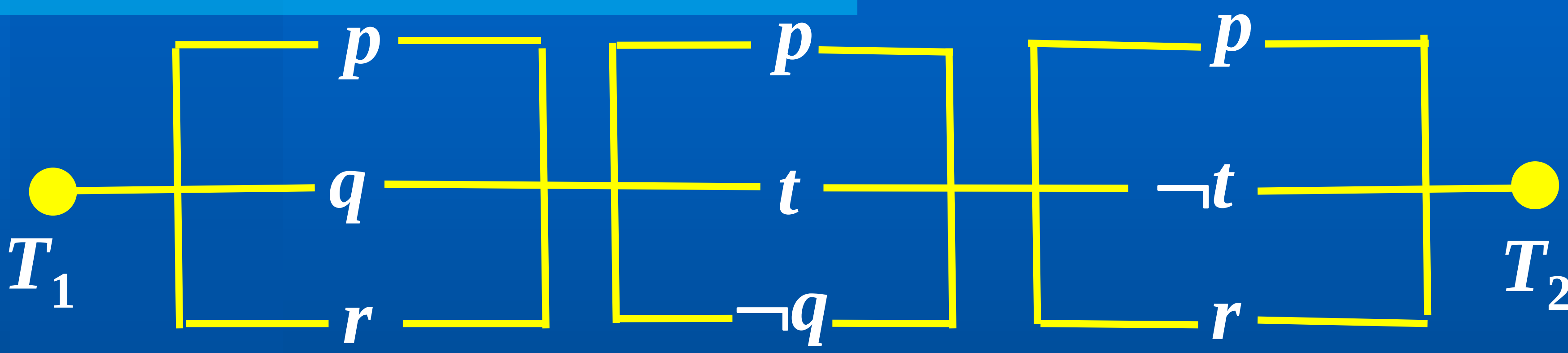
Two independent serial switches $p \wedge q$



Two independent parallel switches $p \vee q$

Simplifying switching networks

Example:



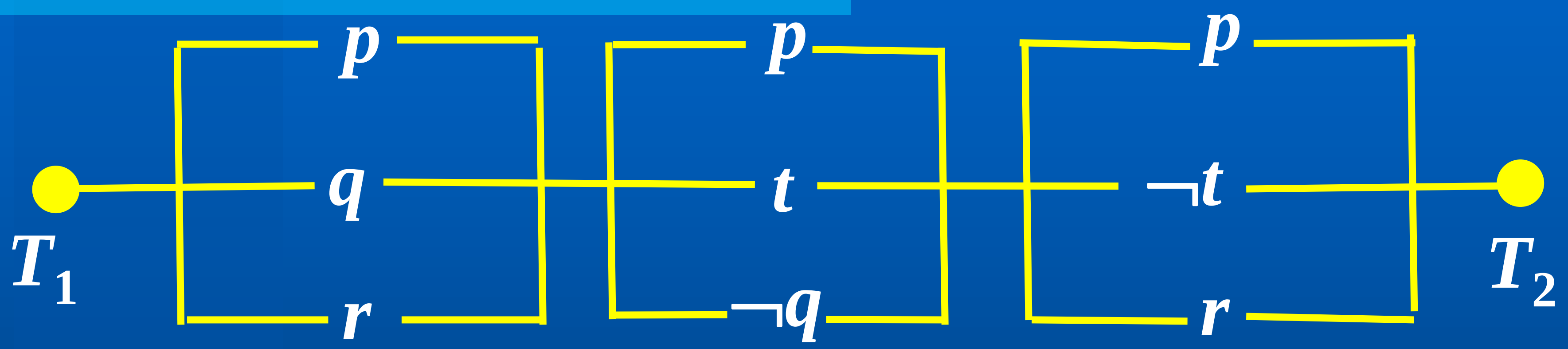
Corresponds to: $(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r)$

One can show (e.g. using Laws of Logic) that this statement is equivalent to:

$p \vee (r \wedge (t \vee \neg q))$ **(cf. book example 2.18)**

Simplifying switching networks

Example (continued):



$$\left(p \vee q \vee r\right) \wedge \left(p \vee t \vee \neg q\right) \wedge \left(p \vee \neg t \vee r\right) \Leftrightarrow p \vee \left(r \wedge \left(t \vee \neg q\right)\right)$$

So the network above can be simplified to: