

Solution – Test Probability Theory 202001233 (CS+BIT M4) - 17 June 2022

- This test contains 8 exercises. A formula sheet and tables of some probability distributions are provided separately.
- Exercises 1-6 are in Part 1 and are either final answer or multiple choice questions. You do not have to show your works for any questions of Part 1.
- Part 2 contains Exercises 7-8. You must show your works to answer these exercises. Also you must motivate your answers where necessary.
- You will write answers on this paper in the designated area. Make sure that your handwriting is legible.
- You must write your name and student ID.
- For each multiple choice question, circle only one option (the option you think is correct). Circling several options will be considered as a wrong answer.
- If you are answering probability questions in percents, then you must write the % sign. For instance, if the answer is 30% and you write just 30, it will be considered as a wrong answer.
- In some exercises we mentioned specifically number of decimal places required. If it is not mentioned specifically, then answer to 3 or 4 decimal places where appropriate.

Please do not write here.

Part 1.

You do not have to show your works for the exercises of Part 1.

Exercise 1. Let A and B be two events such that $0 < P(A) < 1$ and $0 < P(B) < 1$. \bar{A} and \bar{B} represent complements of A and B respectively. For each of the following statements, select either **True** or **False** by drawing a circle around it:

a) A and \bar{A} are independent.	False
b) A is independent of sample space S .	True
c) If A and B are mutually exclusive then $P(\bar{A} \cup \bar{B}) = 1$.	True
d) If A and \bar{B} are independent, then A and B are dependent.	False
e) If $P(A B) = P(B)$ then A and B are independent.	False

Exercise 2. According to a survey, in a state of the USA 54% of all workers have a workplace retirement plan, 69% have health insurance, and 86% have at least one of the benefits. A worker is selected at random.

a) What is the probability he has both health insurance and a retirement plan?

b) The events that “worker having retirement plan” and “worker having health insurance” are:

- A. independent but not disjoint,
- B. both independent and disjoint,
- C. neither disjoint nor independent,
- D. disjoint but not independent.

Circle the correct answer: C

Exercise 3. Let X be a discrete random variable taking values with range $S_X = \{1, 2, \dots, 10\}$.

It is also known that: $P(X \leq 4) = 0.3$, and $P(X \geq 4) = 0.94$.

Find $P(X = 4) =$

Exercise 4. The joint probability function of random variables X and Y is given in the following table:

x	y		
	1	2	3
0	0.1	0.1	0
1	0.2	0.1	0.3
2	0.1	0	0.1

a) $P(X = Y) =$

0.2

b) $P(X < Y) =$

0.7

c) $E(X) =$

1

Exercise 5. In a box there are 10 light bulbs of which 6 are working and 4 are defective. We draw 5 light bulbs at random and let X be the number of defective bulbs selected. Answer the following questions to 3 decimal places.

a) If the draw was done *with replacement*, then $P(X = 2) =$

0.346

b) If the draw was done *without replacement*, then $P(X = 2) =$

0.476

Exercise 6. Let X and Y be two random variables. Which of the following statements is true?

A. $P(X > 5 \text{ and } Y > 5) = P(X + Y > 10)$ is always true.

B. $P(X > 5 \text{ and } Y > 5) \geq P(X + Y > 10)$ is always true.

C. $P(X > 5 \text{ and } Y > 5) \leq P(X + Y > 10)$ is always true.

D. One cannot decide about A, B and C without further information about probability distributions of X and Y .

Circle the correct answer: C

Part 2

You must show your works for the exercises of Part 2. You must also motivate your answers where necessary.

Exercise 7. One needs to score at least 55 out of a total score of 100 to pass a certain math exam. It is noticed that the distribution of the exam scores is roughly symmetric, bell-shaped. The probability distribution of the exam scores is modelled by a normal distribution with mean 64 and standard deviation 10.

- If X is the score of a randomly selected student who took the math exam. What is the probability that the student passed the exam? *i.e.*, determine $P(X \geq 55)$.
- The top 2% student will receive an award. What is the minimum score required to receive the award (answer to one decimal place)? *i.e.* find c such that $P(X \geq c) = 0.02$.
- If we select 5 students at random, what is the probability that at least one of the selected students won the award? You may assume that their scores are independent of each other.
- If we select 5 students at random, what is the probability that their average score is below 55? You may assume that their scores are independent of each other.

Solution. a) Here $X \sim N(64, 10^2)$, and so $Z = \frac{X - \mu}{\sigma} = \frac{X - 64}{10} \sim N(0, 1)$. Hence

$$\begin{aligned} P(X \geq 55) &= P\left(\frac{X - 64}{10} \geq \frac{55 - 64}{10}\right) = P(Z \geq -0.9) = P(Z \leq 0.9) \quad [\text{by symmetry}] \\ &= \Phi(0.9) = 0.8159 \quad (81.59\%). \end{aligned}$$

b) Here $0.02 = P(X \leq c) = P\left(Z = \frac{X - 64}{10} \geq \frac{c - 64}{10}\right) = 1 - \Phi\left(\frac{c - 64}{10}\right) \Rightarrow \Phi\left(\frac{c - 64}{10}\right) = 0.98$. From standard normal table $\Phi(2.05) \approx 0.98$. Thus

$$\frac{c - 64}{10} \approx 2.05 \Rightarrow c \approx 64 + 10 \times 2.05 = 84.5.$$

So one must get at least 84.5 to receive the award.

c) We notice that $P(\text{a student does not receive the award}) = 1 - 0.02 = 0.98$.

$$\begin{aligned} P(\text{at least one student received awards}) &= 1 - P(\text{none of the five students received award}) \\ &= 1 - 0.98 \times 0.98 \times 0.98 \times 0.98 \times 0.98 \quad [\text{by independence}] \\ &= 1 - (0.98)^5 = 0.0961. \quad (9.61\%). \end{aligned}$$

d) If X_i is the score of the i^{th} student, $i = 1, \dots, 5$, then the average score is $\bar{X} = \frac{1}{5} \sum_{i=1}^5 X_i$.

As for each i , $X_i \sim N(64, 10^2)$, so

$$\bar{X} \sim N\left(\mu_{\bar{X}}, \sigma_{\bar{X}}^2\right) = N\left(\mu, \frac{\sigma^2}{n}\right) = N\left(64, \frac{10^2}{5}\right) = N(64, 20) \Rightarrow Z = \frac{\bar{X} - 64}{\sqrt{20}} \sim N(0, 1).$$

$$\begin{aligned} \text{Therefore, } P(\bar{X} \leq 55) &= P\left(\frac{\bar{X} - 64}{\sqrt{20}} \leq \frac{55 - 64}{\sqrt{20}}\right) = P(Z \leq -2.01) = P(Z \geq 2.01) \quad [\text{by symmetry}] \\ &= 1 - \Phi(2.01) = 1 - 0.9778 = 0.0222 \quad (2.22\%). \end{aligned}$$

□

Exercise 8. I went to a vaccination centre to get vaccinated against Corona. Let the random variable X represent the service time (in minutes) at any single counter. The cdf of X is given by

$$F_X(x) = \begin{cases} 0, & \text{if } x \leq 1, \\ 1 - x^{-3}, & \text{if } x > 1. \end{cases}$$

- a) Derive the density function $f_X(x)$ of X .
- b) Compute $\text{Var}(X)$.
- c) When I arrive at the vaccination centre, I find two counters were open, but both of them are busy. I am the first in the queue and will be served as soon as one of the counters becomes free. Let X_1 and X_2 be the service times in counter 1 and 2 respectively. Thus my waiting time to be served is $Y = \min\{X_1, X_2\}$. Here, each of X_1 and X_2 has the same cdf $F_X(x)$ and it is reasonable to assume that X_1 and X_2 are independent. Determine the probability that I have to wait more than 2 minutes, *i.e.*, derive $P(Y > 2)$.

Solution. a) Density of X is

$$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} \frac{d}{dx} 0, & \text{if } x \leq 1, \\ \frac{d}{dx} (1 - x^{-3}), & \text{if } x > 1. \end{cases} = \begin{cases} 0, & \text{if } x \leq 1, \\ 3x^{-4}, & \text{if } x > 1. \end{cases}$$

b)

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f_X(x) dx = \int_1^{\infty} x \cdot (3x^{-4}) dx \\ &= 3 \lim_{b \rightarrow \infty} \int_1^b x^{-3} dx = 3 \lim_{b \rightarrow \infty} \left[\frac{x^{-2}}{-2} \right]_1^b = 3 \lim_{b \rightarrow \infty} \left(\frac{1}{2} - \frac{b^{-2}}{2} \right) = \frac{3}{2}. \end{aligned}$$

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_1^{\infty} x^2 \cdot (3x^{-4}) dx \\ &= 3 \lim_{b \rightarrow \infty} \int_1^b x^{-2} dx = 3 \lim_{b \rightarrow \infty} \left[\frac{x^{-1}}{-1} \right]_1^b = 3 \lim_{b \rightarrow \infty} (1 - b^{-1}) = 3. \end{aligned}$$

$$\text{Var}(X) = E(X^2) - \{E(X)\}^2 = 3 - \left(\frac{3}{2}\right)^2 = 0.75.$$

c)

$$\begin{aligned} P(Y > 2) &= P(\min\{X_1, X_2\} > 2) = P(X_1 > 2, \text{ and } X_2 > 2) \\ &= P(X_1 > 2) \cdot P(X_2 > 2) \quad \text{[by independence]} \\ &= [1 - F_X(2)] \cdot [1 - F_X(2)] = (1 - (1 - 2^{-3}))^2 \\ &= 2^{-6} = 0.0156 \quad (1.56\%). \end{aligned}$$

□