Test Probability Theory 202001233 (CS+BIT M4) - 17 June 2022

- This test contains 8 exercises. A formula sheet and tables of some probability distributions are provided separately.
- Exercises 1-6 are in Part 1 and are either final answer or multiple choice questions. You do not have to show your works for any questions of Part 1.
- Part 2 contains Exercises 7-8. You must show your works to answer these exercises. Also you must motivate your answers where necessary.
- For each multiple choice question, circle only one option (the option you think is correct). Circling several options will be considered as a wrong answer.
- If you are answering probability questions in percents, then you must write the % sign. For instance, if the answer is 30% and you write just 30, it will be considered as a wrong answer.
- You will write answers on this paper in the designated area. Make sure that your handwriting is legible.
- You must write your name and student ID.
- In some exercises we mentioned specifically number of decimal places required. If it is not mentioned specifically, then answer to 3 or 4 decimal places where appropriate.

Please do not write here.

Part 1.

You do not have to show your works for the exercises of Part 1.

Exercise 1. Let A and B be two events such that $0 < P(A) < 1$ and $0 < P(B) < 1$. $\overline{A}$ and $\overline{B}$ represent complements of A and B respectively. For each of the following statements, select either **True** or **False** by drawing a circle around it:

a)	A and $\overline{A}$ are independent.	**True**	**False**
b)	A is independent of sample space S.	**True**	**False**
c)	If A and B are mutually exclusive then $P(\overline{A} \cup \overline{B}) = 1$.	**True**	**False**
d)	If A and $\overline{B}$ are independent, then A and B are dependent.	**True**	**False**
e)	If $P(A \mid B) = P(B)$ then A and B are independent.	**True**	**False**

Exercise 2. According to a survey, in a state of the USA 54% of all workers have a workplace retirement plan, 69% have health insurance, and 86% have at least one of the benefits. A worker is selected at random.

a) What is the probability that the selected worker has health insurance but no retirement plan?

b) The events that "worker having retirement plan" and "worker having health insurance" are:

A. independent but not disjoint,

B. both independent and disjoint,

C. neither disjoint nor independent,

D. disjoint but not independent.

Circle the correct answer: A B C D

Exercise 3. Let X be a discrete random variable taking values with range $S_X = \{1, 2, \ldots, 10\}$.
It is also known that: $P(X \leq 4) = 0.3$, and $P(X \geq 4) = 0.94$.

Find $P(X = 4) =$

Exercise 4. The joint probability function of random variables X and Y is given in the following table:

	y		
x	1	2	3
0	0.1	0.1	0
1	0.2	0.1	0.3
2	0.1	0	0.1

Determine the following:

a) $P(X = Y) =$

b) $P(X < Y) =$

c) $E(X) =$

Exercise 5. In a box there are 10 light bulbs of which 6 are working and 4 are defective. We draw 5 light bulbs at random and let X be the number of defective bulbs selected. *Answer the following questions to 3 decimal places.*

a) If the draw was done *with replacement*, then $P(X = 2) =$

b) If the draw was done *without replacement*, then $P(X = 2) =$

Exercise 6. Let X and Y be two random variables. Which of the following statements is true?

A. $P(X > 5 \text{ and } Y > 5) = P(X + Y > 10)$ is always true.

B. $P(X > 5 \text{ and } Y > 5) \geq P(X + Y > 10)$ is always true.

C. $P(X > 5 \text{ and } Y > 5) \leq P(X + Y > 10)$ is always true.

D. One cannot decide about A, B and C without further information about probability distributions of X and Y.

Circle the correct answer: A B C D

Part 2

You must show your works for the exercises of Part 2. You must also motivate your answers where necessary.

Exercise 7. One needs to score at least 55 out of a total score of 100 to pass a certain math exam. It is noticed that the distribution of the exam scores is roughly symmetric, bell-shaped. The probability distribution of the exam scores is modelled by a normal distribution with mean 64 and standard deviation 10.

a) If X is the score of a randomly selected student who took the math exam. What is the probability that the student passed the exam? *i.e.*, determine $P(X \geq 55)$.

b) The top 2% student will receive an award. What is the minimum score required to receive the award (answer to one decimal place)? *i.e.* find c such that $P(X \geq c) = 0.05$.

c) If we select 5 students at random, what is the probability that at least one of the selected students won the award? You may assume that their scores are independent of each other.

d) If we select 5 students at random, what is the probability that their average score is below 55? You may assume that their scores are independent of each other.

(Continue with answering Exercise 7)

Exercise 8. I went to a vaccination centre to get vaccinated for Corona. Let the random variable X represent the service time (in minutes) at any single counter. The cdf of X is given by

$$F_X(x) = \begin{cases} 0, & \text{if } x \leq 1, \\ 1 - x^{-3}, & \text{if } x > 1. \end{cases}$$

a) Derive the density function $f_X(x)$ of X.

b) Compute $\mathrm{Var}(X)$.

c) When I arrive at the vaccination centre, I find two counters were open, but both of them are busy. I am the first in the queue and will be served as soon as one of the counters becomes free. Let X_1 and X_2 be the service times in counter 1 and 2 respectively. Thus my waiting time to be served is $Y = \min\{X_1, X_2\}$. Here, each of X_1 and X_2 has the same cdf $F_X(x)$ and it is reasonable to assume that X_1 and X_2 are independent. Determine the probability that I have to wait more than 2 minutes, *i.e.*, derive $P(Y > 2)$.

(Continue with answering Exercise 8)