Solutions Exersises Chapter 8

1. **a**. X is exponentially distributed with parameter $\lambda = \frac{1}{60}$, so (see properties in section 8.1):

$E(X) = \frac{1}{\lambda} = 60$ and because the distribution is memoryless: $P(X \geq 30) = P(X \geq 90|X \geq 60) = e^{-\frac{1}{2}}$.

b. $P(N = n) = P[30(n-1) < X \leq 30n] = P\big(X > 30(n-1)\big) - P(X > 30n)$

$$= e^{-\lambda\cdot 30(n-1)} - e^{-\lambda\cdot 30n} = e^{-30\lambda\cdot(n-1)}\left(1 - e^{-30\lambda}\right) = \left[e^{-30\lambda}\right]^{n-1}\left(1 - e^{-30\lambda}\right), n = 1,2,\ldots$$

N is apparantly geometrically distributed with parameter $p = 1 - e^{-30\lambda} = 1 - e^{-0.5} \approx 0.393$

c. The costs of one call is $15N$, so $E(15N) = 15E(N) = 15 \cdot \frac{1}{p} \approx 38.12\ cent$ (> 30 $cent$)

d. $var(15N) = 15^2 \cdot var(N) = 225 \cdot \frac{1-p}{p^2} \approx 881.48\ cent^2$

2. **a**. The event $\{Y > t\}$ occurs if in the interval of time no arrivals take place, so $\{X = 0\}$.

Then: $P(Y > t) = P(X = 0)$.

b. $P(X = 0) = \frac{(at)^0 e^{-at}}{0!} = e^{-at}$, since X is Poisson distributed with $\mu = at$

c. $P(Y > t) = e^{-at}$ is the "survival probability" for an exponential distribution with $\lambda = a$.

($P(Y > t) = e^{-at}$ can be used to compute $F_Y(t) = 1 - P(Y > t) = 1 - e^{-at}$, so $f_Y(t) = \frac{d}{dy}F_Y(t) = ae^{-at}$ ($t > 0$), the exponential density function with parameter a.)

3. **a.** From $Y = \ln\left(\frac{1}{X}\right)$ it follows that for $y > 0$:

$$F_Y(y) = P\left(ln\left(\frac{1}{X}\right) \leq y\right) = P\left(\frac{1}{X} \leq e^y\right) = P\left(X \geq \frac{1}{e^y}\right) = 1 - F_X(e^{-y})$$

So $f_Y(y) = \frac{d}{dy}F_Y(y) = -f_X(e^{-y}) \cdot -e^{-y} = +e^{-y}f_X(e^{-y})$.

Since $f_X(x) = 1$ for $0 \leq x \leq 1$, we have: $f_Y(y) = e^{-y} \cdot 1$ if $0 \leq e^{-y} \leq 1$, so if $y \geq 0$.

Conclusion: $Y = \ln\left(\frac{1}{X}\right)$ is exponentially distributed with parameter λ = 1.

b. $E(Y) = \frac{1}{\lambda} = 1$ and $E(X) = \frac{1}{2}$, so $1 = E(Y) \neq \ln\left(\frac{1}{EX}\right) = \ln(2)$

4. **a**. Given is that every $X_i \sim Exp(\lambda)$, where $E(X_i) = 12 = \frac{1}{\lambda}$, so $\lambda = \frac{1}{12}$.

$var(X_i) = \frac{1}{\lambda^2} = 144$ and $E(\sum_{i=1}^{6} X_i) = 6 \cdot E(X_i) = 6 \cdot 12 = 72$.

b. Because of the lack of memory property: $P(X_1 > 15|X_1 > 3) = P(X_1 > 12) = e^{-\frac{1}{12}\cdot 12} = e^{-1}$.

c. The expected waiting time is 12 sec., so the expected number of arrivals is $\frac{60}{12} = 5$, or:

$N \sim Poisson(\lambda t)$, where $\lambda t = \frac{1}{12} \cdot 60 = 5$, so $P(N \geq 6) = 1 - P(X \leq 5) = 38.4\%$.

d. $\sum_{i=1}^{6} X_i$ has an Erlang-distribution with $n = 6$ and $\lambda = \frac{1}{12}$.

The event $\{\sum_{i=1}^{6} X_i \leq 60\}$ implies that 6 or more arrivals take place in $[0, 60]$, so:

$$P\left(\sum_{i=1}^{6} X_i \leq 60\right) = P(N \geq 6) = 38.4\%$$

5.

a. According to the presented theory the sum of independent, $Exp\left(\frac{1}{4}\right)$-distributed is Erlang distributed:

S_n has an Erlang distribution with parameters n and λ = ¼ .

$E(S_n) = \frac{n}{\lambda} = 4n$ and $var(S_n) = \frac{n}{\lambda^2} = 16n$.

b. see example 7.2.2 (applying the convolution-integral:)

$f_{X_1+X_2}(z) = \int_{-\infty}^{\infty} f_{X_1}(x) f_{X_2}(z-x)dx$, where $f_{X_1}(x) = 0$, if $x < 0$ and $f_{X_2}(z-x) = 0$, if $x > z$

$= \int_0^z \lambda e^{-\lambda x}\lambda e^{-\lambda(z-x)}dx$ $\left(\text{met } \lambda = \frac{1}{4}\right)$

$= \int_0^z \lambda^2 e^{-\lambda z}dx$ Notice that $\lambda^2 e^{-\lambda z}$ is a constant in the integration w.r.t. x.

$$= \lambda^2 e^{-\lambda z} \cdot x\Big|_{x=0}^{x=z}$$

$$= \lambda^2 z e^{-\lambda z}, \text{ for } z \geq 0$$

And $f_{X_1+X_2}(z) = 0$, if $z < 0$.

c. $P(\overline{X}_2 > 5) = P\left(\frac{S_2}{2} > 5\right) = P(S_2 > 10) = \int_{10}^{\infty} \lambda^2 \mathrm{z} \mathrm{e}^{-\lambda \mathrm{z}} \mathrm{dz} = \cdots \textit{partial integration} .. = \frac{7}{2} e^{-\frac{3}{2}}$

d. For large n we can apply the CLT: $\overline{X}_n$ is approximately $N\left(\mu, \frac{\sigma^2}{n}\right)$, where in this case $n = 100$, $\mu = \frac{1}{\lambda} = 4$ and $\sigma^2 = \frac{1}{\lambda^2} = 16$, so:

$$P(\overline{X}_{100} > 5) \overset{\text{CLT}}{\approx} P\left(Z > \frac{5-4}{\sqrt{\frac{16}{100}}}\right) = P(Z > 2.5) = 1 - \Phi(2.5) = 0.62\%$$