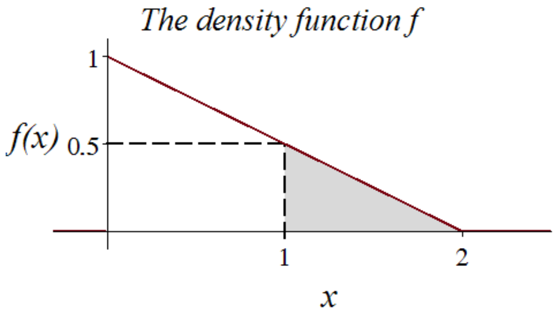
Solutions of exercises chapter 6



1. **a.** $P(X > 1) = \int_1^\infty f(x)dx = \int_1^2 \left(1 - \frac{1}{2}x\right) dx$

$= x - \frac{1}{4}x^2\Big|_{x=1}^{x=2} = 1 - \frac{3}{4} = \frac{1}{4},$

(or determine the shaded area of the triangle: $\frac{1}{2} \cdot 1 \cdot \frac{1}{2} = \frac{1}{4}$)

b. $E(X) = \int_0^2 x \cdot \left(1 - \frac{1}{2}x\right) dx = \frac{1}{2}x^2 - \frac{1}{6}x^3\Big|_{x=0}^{x=2} = \frac{2}{3}$

$E(X^2) = \int_0^2 x^2 \cdot \left(1 - \frac{1}{2}x\right) dx = \frac{1}{3}x^3 - \frac{1}{8}x^4\Big|_{x=0}^{x=2} = \frac{2}{3}$

$var(X) = E(X^2) - (EX)^2 = \frac{2}{3} - \left(\frac{2}{3}\right)^2 = \frac{2}{9}$

c. $F(x) = P(X \le x)$, so $F(x) = 0$, if $x < 0$ and $F(x) = 1$, if $x > 2$.

if $0 \le x \le 2$, then $F(x) = \int_0^x \left(1 - \frac{1}{2}u\right) du = u - \frac{1}{4}u^2\Big|_{u=0}^{u=x} = x - \frac{1}{4}x^2$

So $P(X > 1) = 1 - F(1) = 1 - \frac{3}{4} = \frac{1}{4}$

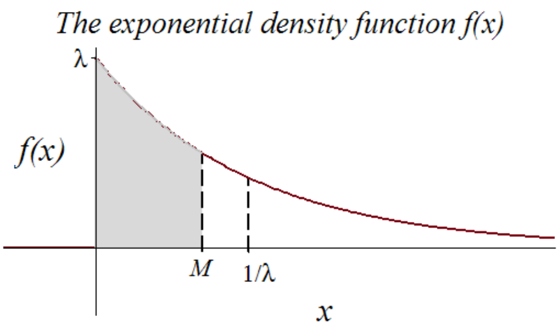
2. **a**. $f(x) = \lambda e^{-\lambda x}$, if $x \ge 0$ and $f(x) = 0$, if $x < 0$.

$\int_{-\infty}^\infty f(x)\, dx = \int_0^\infty \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}|_0^\infty = -0 - (-1) = 1$ (graph: see below).

b. Use partial integration (see the appendix Mathematical Techniques):

$E(X) = \int_0^\infty x \cdot \lambda e^{-\lambda x}\, dx = x \cdot -e^{-\lambda x}|_0^\infty + \int_0^\infty e^{-\lambda x}\, dx = -\frac{1}{\lambda} e^{-\lambda x}|_0^\infty = +\frac{1}{\lambda}$

$P(X > EX) = \int_{1/\lambda}^\infty \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}|_{\frac{1}{\lambda}}^\infty = e^{-\lambda \cdot \frac{1}{\lambda}} = e^{-1} \approx 36.8\% \; (< \frac{1}{2})$



c. $P(X > M) = \int_M^\infty \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}|_M^\infty = e^{-\lambda M} = \frac{1}{2}$, so $M = \frac{\ln(2)}{\lambda}$

d. The mode = 0 (see graph)

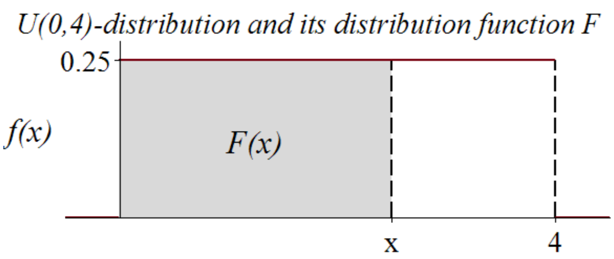
3. **a.** - $f(x) = \frac{1}{4}$, for $0 < x < 4$:

$\int_{-\infty}^\infty f(x)\, dx = 4 \cdot \frac{1}{4} = 1$

- $E(X) = 2 =$ median (because of symmetry of f.

So $P(X > EX) = P(X \le M) = \frac{1}{2}$

- The mode can be each value of the interval (0,4).

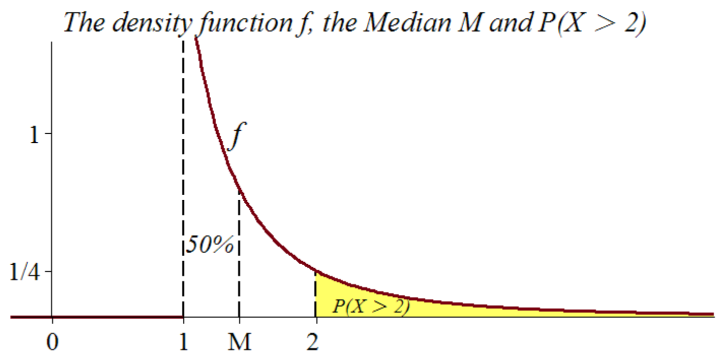


b. $F_X(x) = \begin{cases} 0 & \text{if} \quad x \le 0 \\ \frac{x}{4} & \text{if } 0 < x < 4 \\ 1 & \text{if} \quad x \ge 4 \end{cases}$

4. a. $f(x) = \frac{c}{x^3}$ if $x > 1$, so $\int_{-\infty}^{\infty} f(x)dx = \int_1^{\infty} \frac{c}{x^3}\, dx = c \cdot -\frac{1}{2}x^{-2}\Big|_1^{x\to\infty} = 0 + c \cdot \frac{1}{2} = 1$

So $c = 2$.

$P(X > 2) = \int_2^{\infty} \frac{2}{x^3}\, dx = -x^{-2}|_{x=2}^{x\to\infty} = 0 - (-2^{-2}) = \frac{1}{4}$



b. $E(X) = \int_{-\infty}^{\infty} f(x)dx = \int_1^{\infty} x \cdot \frac{2}{x^3}\, dx = -2x^{-1}|_1^{x\to\infty} = 0 + 2 = 2$

The median M: $P(X \ge M) = \int_M^{\infty} \frac{2}{x^3}\, dx = -x^{-2}|_{x=M}^{x\to\infty} = 0 + \frac{1}{M^2} = \frac{1}{2}$, so $M = \sqrt{2}$

c. $F(x) = \begin{cases} 0 & \text{if } x < 1 \\ \int_1^x \frac{2}{u^3}\, du = -u^{-2}|_{u=1}^{u=x} = 1 - \frac{1}{x^2} & \text{, if } x \ge 1 \end{cases}$

Check: $(x > 1)$: $f(x) = \frac{d}{dx}F(x) = -\left(-\frac{2}{x^3}\right) = \frac{2}{x^3}$ (correct).

5. a. 1. $F_Y(y) = P(5 - 2X \le y) = P(-2X \le y - 5) = P\left(X \ge \frac{y-5}{-2}\right) = 1 - F_X\left(-\frac{y-5}{2}\right)$

2. $f_Y(y) = \frac{d}{dy}F_y(y) = \frac{1}{2}f_X\left(\frac{y-5}{-2}\right)$

3. $f_X\left(\frac{y-5}{-2}\right) = 1$, if $0 < \frac{y-5}{-2} < 1$, so if $3 < y < 5 \to f_Y(y) = \frac{1}{2} \cdot 1$, for $3 < y < 5$

So $Y \sim U(3, 5)$

b. Choose $Y = a + (b - a) \cdot X$

c. $Y = 2X$, if $X \sim Exp(\lambda = 3)$:

1. $F_Y(y) = P(2X \le y) = P\left(X \le \frac{1}{2}y\right) = F_X\left(\frac{1}{2}y\right)$

2. $f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X\left(\frac{1}{2}y\right) = \frac{1}{2} \cdot f_X\left(\frac{1}{2}y\right)$

3. Since $f_X(x) = 3e^{-3x}$ for $x \ge 0$, we have for $\frac{1}{2}y \ge 0$, or $y \ge 0$:

$f_Y(y) = \frac{1}{2} \cdot f_X\left(\frac{1}{2}y\right) = \frac{1}{2} \cdot 3e^{-3\cdot\frac{1}{2}y} = \frac{3}{2}e^{-\frac{3}{2}y}$. So $Y \sim \exp\left(\frac{3}{2}\right)$

$Z = X^2$, if $X \sim Exp(\lambda = 3)$:

1. $F_Z(z) = P(X^2 \le z) = P(-\sqrt{z} \le X \le \sqrt{z}) = F_X(\sqrt{z}) - F_X(-\sqrt{z})$, if $z > 0$

2. $f_Z(z) = \frac{d}{dy}[F_X(\sqrt{z}) - F_X(-\sqrt{z})] = \frac{1}{2\sqrt{z}} \cdot f_X(\sqrt{z}) + \frac{1}{2\sqrt{z}} \cdot f_X(-\sqrt{z})$. $(z > 0)$

3. Since $f_X(x) = 3e^{-3x}$ for $x \geq 0$, we have $f_X(\sqrt{z}) = 3e^{-3\sqrt{z}}$ and $f_X(-\sqrt{z}) = 0$, so

$$f_Z(z) = \frac{1}{2\sqrt{z}} \cdot 3e^{-3\sqrt{z}} + \frac{1}{2\sqrt{z}} \cdot 0 = \frac{1}{2\sqrt{z}} \cdot 3e^{-3\sqrt{z}} \text{ , for } z > 0.$$

(This is not a regular distribution, but one can verify that it defines a density function: $f_Z(z) \geq 0$ and

$\int_{-\infty}^{\infty} f_Z(z)dz = \int_0^{\infty} \frac{1}{2\sqrt{z}} \cdot 3e^{-3\sqrt{z}}\, dz$ (use a substitution: $x = \sqrt{z}$, so $dx = \frac{1}{2\sqrt{z}} dz$)

$= \int_0^{\infty} 3e^{-3x}\, dx$

$= -e^{-3x}|_{x=0}^{x\to\infty} = 0 + 1 = 1$)

6. a. 1. $F_Y(y) = P\left(\frac{1}{X} \leq y\right) = P\left(X \geq \frac{1}{y}\right)$

$= 1 - F_X\left(\frac{1}{y}\right)$, $y > 0$

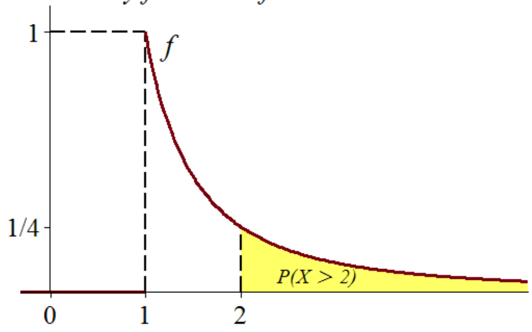
(And $F_Y(y) = P\left(\frac{1}{X} < y\right) = 0$ if $y < 0$)

2. $f_Y(y) = \frac{d}{dy} F_y(y) = -\frac{1}{y^2} \cdot -f_X\left(\frac{1}{y}\right)$

3. $f_X\left(\frac{1}{y}\right) = 1$, if $\frac{1}{y} > 0$, so if $y > 1$

So $f_Y(y) = \frac{1}{y^2} \cdot 1 = \frac{1}{y^2}$ if $y > 1$

The density function of Y: EY does not exist



b. $P(Y > 2) = \int_2^{\infty} f_Y(y)\, dy = \int_2^{\infty} \frac{1}{y^2} dy = -y^{-1}|_2^{\infty} = \frac{1}{2}$

$P(Y > 2) = P\left(\frac{1}{X} > 2\right) = P\left(X < \frac{1}{2}\right) = \frac{1}{2}$

c. $E(Y) = \int_{-\infty}^{\infty} y f_Y(y)dy = \int_1^{\infty} y \cdot \frac{1}{y^2} dy = \ln(y)\,|_1^{\infty} = \infty$, so $E(Y)$ **does not exist**.

Likewise $E(Y) = E\left(\frac{1}{X}\right) = \int_{-\infty}^{\infty} \frac{1}{x} f_X(x)dx = \int_0^1 \frac{1}{x} dy = \ln(x)\,|_0^1$ does not exist.

7. a. if $y > 0$ we have: $F_Y(y) = P\left(\sqrt{|X|} \leq y\right) = P(|X| \leq y^2) = P(-y^2 \leq X \leq y^2)$

$= F_X(y^2) - F_X(-y^2)$

Since $F_X(x) = 1 - e^{-x}$, if $x \geq 0$ and $F_X(x) = 0$, if $x < 0$,

$F_Y(y) = \left(1 - e^{-y^2}\right) - 0 = 1 - e^{-y^2}$, for $y > 0$.

b. $f_Y(y) = \frac{d}{dy} F_y(y) = 2ye^{-y^2}$, for $y > 0$

$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)dy = \int_0^{\infty} y \cdot 2ye^{-y^2} dy = \ldots$ partial integration $\ldots = \int_0^{\infty} e^{-y^2} dy$,

Here we can apply the standard normal distribution, so $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 1$,

Use the substitution $x = \sqrt{2} \cdot y$, to find $\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$, so $E(Y) = \frac{1}{2}\sqrt{\pi}$

8. a. $P(X > 2) = P\left(\frac{X-1}{2} > \frac{2-1}{2}\right)$

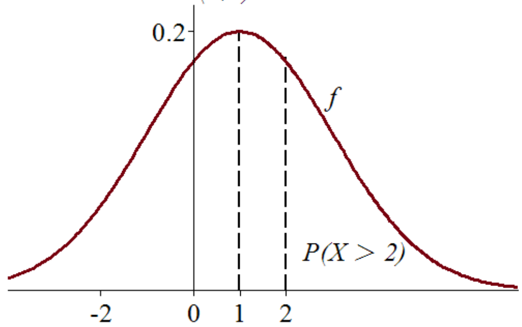
$= P\left(Z > \frac{1}{2}\right) = 1 - \Phi(0.5) = 0.3085.$

$P(|X| > 2) = P(X > 2) + P(X < -2)$

$= P\left(Z > \frac{1}{2}\right) + P\left(Z < \frac{-2-1}{2}\right)$

$= 1 - \Phi(0.5) + \Phi(-1.5)$

The N(1,4)-distribution



$= 0.3085 + (1 - 0.9332) = 0.3753$

$P(|X-1| < 2) = P(-2 < X-1 < +2)$

$= P\left(-\frac{2}{2} < \frac{X-1}{2} < \frac{2}{2}\right)$

$= \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 2 \cdot 0.8417 - 1 = 68.34\%.$

b. $P(X \le c) = P\left(\frac{X-1}{2} \le \frac{c-1}{2}\right) = \Phi\left(\frac{c-1}{2}\right) = 90\%$, so $\frac{c-1}{2} = 1.28$.

$c = 1 + 2 \cdot 1.28 = 3.56$ is the 90^{th} percentile of X.

c. $P(X \le c) = P\left(\frac{X-1}{2} \le \frac{c-1}{2}\right) = \Phi\left(\frac{c-1}{2}\right) = 10\%$, so $\frac{c-1}{2} = -1.28$.

$c = 1 - 2 \cdot 1.28 = -1.56$ is the 10^{th} percentile of X.

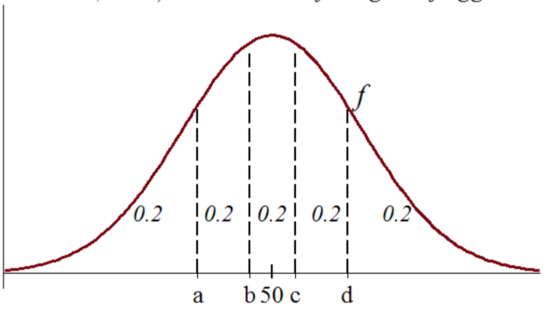
9. For instance: $P(-2\sigma < X - \mu < 2 \cdot \sigma) = P\left(-2 < \frac{X-\mu}{\sigma} < 2\right) = \Phi(2) - \Phi(-2)$

$= 2\Phi(2) - 1 = 2 \cdot 0.9772 - 1 = 0.9544 \approx 95.4\%$

And: $P(-\sigma < X - \mu < \sigma) = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 2 \cdot 0.8413 - 1 \approx 68.3\%$

$P(-3\sigma < X - \mu < 3\sigma) = \Phi(3) - \Phi(-3) = 2\Phi(3) - 1 = 2 \cdot 0.9987 - 1 = 99.68\%$

The N(50,25)-distribution of weights of eggs



10. The interval bounds a, b, c and d are such that b and c are symmetric about 50, as are a and d.

$P(X \le c) = 0.60$, so $P\left(Z \le \frac{c-50}{5}\right) = \Phi\left(\frac{c-50}{5}\right) = 0.60$

In the $N(0,1)$-table we find that $\Phi(z) = 0.60$, if $z \approx 0.25$,

so $\frac{c-50}{5} \approx 0.25$, or $c \approx 50 + 5 \cdot 0.25 = 51.25$ gram and $b \approx 50 - 1.25 = 48.75$ gram.

(*since the probability 0.60 is not found exact exactly in the N(0,1)-table, we could approximate the z-value more precise by linear interpolation:* $z = 0.254$ *and* $c \approx 51.27$ *gram.*)

Similarly:

$P(X \le d) = \Phi\left(\frac{d-50}{5}\right) = 0.80$, so $d \approx 50 + 5 \cdot 0.84 = 54.20$ and $a \approx 45.80$ gram.

11. a. Since $E(X) = \mu$, we have:

$E(X-\mu)^3 = E(X^3 - 3 \cdot X^2 \cdot \mu + 3 \cdot X \cdot \mu^2 - \mu^3)$

$= E(X^3) - 3\mu E(X^2) + 3\mu^2 E(X) - \mu^3 = E(X^3) - 3\mu E(X^2) + 2\mu^3$

b. $E(X) = \frac{1}{2}$, $E(X^2) = \int_0^1 x^2 \cdot dx = \frac{1}{3}x^3|_0^1 = \frac{1}{3}$ and $E(X^3) = \int_0^1 x^3 \cdot dx = \frac{1}{4}x^4|_0^1 = \frac{1}{4}$.

And applying a.: $E(X-\mu)^3 = \frac{1}{4} - 3 \cdot \frac{1}{2} \cdot \frac{1}{3} + 2 \cdot \left(\frac{1}{2}\right)^3 = 0$

(of directly: $E(X-\mu)^3 = \int_0^1 \left(x - \frac{1}{2}\right)^3 dx = \frac{1}{4}\left(x - \frac{1}{2}\right)^4 |_0^1 = \frac{1}{4}\left(\frac{1}{16} - \frac{1}{16}\right) = 0$).

c. $E(X) = \frac{1}{\lambda} = 1$ and since $var(X) = E(X^2) - (EX)^2 = 1$, we have $E(X^2) = 1 + 1 = 2$.

$E(X^3) = \int_0^\infty x^3 \cdot e^{-x} dx = x^3 \cdot -e^{-x}|_0^1 + 3\int_0^\infty x^2 \cdot e^{-x} dx = 3E(X^2) = 6.$

$E(X-\mu)^3 = 6 - 3 \cdot 1 \cdot 2 + 2 \cdot 1 = 2$

d. Correct: the uniform distribution is symmetric: $E(X-\mu)^3 = 0$.

and the exponential distribution is skewed to the right: $E(X-\mu)^3 = +2$.