Solutions Exercises Chapter 4

1. **a**. $P(X > 0) = P(X = 1) + P(X = 3) = 0.4 + 0.1 = 0.5$
b. $E(X) = \sum_x x \cdot P(X = x) = -0.2$ (see the computations in the table below)
c. $E(X^2) = \sum_x x^2 \cdot P(X = x) = 4.6$ (see the table)
d. $var(X) = E(X^2) - (EX)^2 = 4.6 - (-0.2)^2 = 4.56$ and $\sigma_X = \sqrt{var(X)} \approx 2.14$

x	-5	-2	0	1	3	Total
$P(X = x)$	0.1	0.2	0.2	0.4	0.1	1
$x \cdot P(X = x)$	-0.5	-0.4	0	0.4	0.3	$-0.2 = E(X)$
$x^2 \cdot P(X = x)$	2.5	0.8	0	0.4	0.9	$4.6 = E(X^2)$

2. $P(X = k) = \frac{1}{10}, k = 0, 1, 2, \ldots, 9$
$E(X) = 4.5$ (symmetry),
$E(X^2) = \sum_x x^2 \cdot P(X = x) = (0^2 + 1^2 + \cdots + 9^2) \cdot \frac{1}{10} = 28.5$
and $var(X) = E(X^2) - (EX)^2 = 28.5 - 4.5^2 = 8.25$

3. For the profit W we can find the following probabilities (with binomial formula):
$P(W = -1) = \left(\frac{9}{10}\right)^3 = \frac{729}{1000}$
$P(W = 0) = 3 \cdot \frac{1}{10} \cdot \left(\frac{9}{10}\right)^2 = \frac{243}{1000}$
$P(W = 9) = 3 \cdot \left(\frac{1}{10}\right)^2 \cdot \frac{9}{10} = \frac{27}{1000}$
$P(W = c) = \left(\frac{1}{10}\right)^3 = \frac{1}{1000}$ (Check that the sum of all probabilities = 1)
(we used that the number of J's is $B\left(3, \frac{1}{10}\right)$-distributed:
$P(W = 0)$ is for instance the probability of one J)
The profit c for 3 J's must be such that the expected profit is 0:
$E(W) = \sum_w w \cdot P(W = w) = -1 \cdot \frac{729}{1000} + 0 \cdot \frac{243}{1000} + 9 \cdot \frac{27}{1000} + c \cdot \frac{1}{1000} = 0$ if
$c = 729 - 9 \cdot 27 = 486$: so the payment = profit + inlay (1) = 487

4. X = "the number of 6`s" is $B\left(3, \frac{1}{6}\right)$-distributed. So if W is the profit of the player, then:
$P(W = -1) = P(X = 0) = \left(\frac{5}{6}\right)^3 = \frac{125}{216}$
$P(W = 1) = P(X = 1) = 3 \cdot \left(\frac{5}{6}\right)^2 \cdot \frac{1}{6} = \frac{75}{216}$
$P(W = 2) = P(X = 2) = 3 \cdot \frac{5}{6} \cdot \left(\frac{1}{6}\right)^2 = \frac{15}{216}$
$P(W = 3) = P(X = 3) = \left(\frac{1}{6}\right)^3 = \frac{1}{216}$ (check: sum of probabilities = 1)
$E(W) = \sum_w w \cdot P(W = w) = -1 \cdot \frac{125}{216} + 1 \cdot \frac{75}{216} + 2 \cdot \frac{15}{216} + 3 \cdot \frac{1}{216} = 0 - \frac{17}{216} \approx -0.0787$
The expected loss is about 8 Eurocent per game.

5. Remark: the solution below is the result of analysing the problem. This is an ability that is not easy, but it can be trained. Sometimes it is helpful to physically imagine the problem. And often a first step is simplifying the problem: in this case we are asked in the a-part to find the solution if $N = 10$ and $n = 4$ are chosen

a. $P(X = 7)$, the probability that the highest of the four chosen from 1, 2, …, 10 is 7. Sketch it:

1 | 2 | 3 | 4 | 5 | 6	7	8 | 9 | 10
Choose 3 out of 6	highest	none

So $P(X = 7) = \frac{\binom{6}{3}}{\binom{10}{4}}$.

b. The probability above can be generalized into a formula: $P(X = k)$ is the probability that k is the highest number in the sample of n out of N:

***N* marbles in the vase:**	$1, 2, \ldots, k-1$	k	$k+1, \ldots, N$
***n* chosen marbles:**	$n-1$ out of $k-1$	Highest	Not chosen

$P(X = k) = \frac{\binom{k-1}{n-1}}{\binom{N}{n}}, k = n, n+1, \ldots, N$

(Try this approach again in exercise 15).

6. **a**. $P(X \leq 7) = 0.998$ (see table).

b. $P(X \geq 7) = 1 - P(X \leq 6) = 1 - 0.989 = 0.011$

c. If Y is the corresponding **number of failures**, so $Y \sim B(15, 0.4)$, then:
$P(X = 9) = P(Y = 6) = P(Y \leq 6) - P(Y \leq 5) = 0.610 - 0.403 = 20.7\%$
Check: $P(X = 9) = \binom{15}{9} 0.6^9 0.4^6 \approx 20.7\%$. Correct.

d. $P(X < 12) = P(Y > 3) = 1 - P(Y \leq 3) = 1 - 0.091 = 0.909$

7. **a**. $P(X = 5) = P(X \leq 5) - P(X \leq 4) = 0.916 - 0.815 = 0.101$
Check: $P(X = 5) = \frac{3^5}{5!} \cdot e^{-3} \approx 0.101$. Correct.

b. $P(X < 2) = P(X \leq 1) = 0.199$.

c. $P(X > 3) = 1 - P(X \leq 3) = 1 - 0.647 = 0.353$

8.

a. The number of events that occur in the interval is Poisson distributed, with mean $\mu = \frac{30}{6}$ per ten minutes: $P(X = 2) = \frac{5^2}{2!} e^{-5} \approx 0.0842$. $E(X) = \mu = 5$.

b. Random draws without replacement from a dichotomous population of 3 men and 2 women: use the hypergeometric distribution.
$P(X = 2) = \frac{\binom{2}{2}\binom{3}{0}}{\binom{5}{2}} = \frac{1}{10}$ and $EX = n \cdot \frac{R}{N} = 2 \cdot \frac{2}{5} = \frac{4}{5}$

c. Large population, so without replacement is approximately the same as with replacement: 100 Bernoulli-experiments with success probability $p = 0.02$.
$X \sim B(100, 0.02)$
$P(X = 2) = \binom{100}{2} 0.02^2 0.98^{98} \approx 27.3\%$ and $E(X) = np = 2$.
In this case p is small $P(X = 2)$, so X can be approximated with the Poisson distribution, since $\mu = np = 2$ is small (< 5) and n is large (> 25):
$P(X = 2) \approx \frac{2^2}{2!} e^{-2} \approx 27.1\%$

d. Every trial has a success probability $\frac{1}{10}$ and the trials are independent: we are repeating Bernoulli-trials until a success occurs, so:

$X \sim geometric\left(p = \frac{1}{10}\right)$

$P(X = 2) = \frac{9}{10} \cdot \frac{1}{10} = 0.09$ and $E(X) = \frac{1}{p} = 10$

e. $P(X = 1) = \frac{1}{10}, P(X = 2) = \frac{9}{10} \cdot \frac{1}{9} = \frac{1}{10}, P(X = 3) = \frac{9}{10} \cdot \frac{8}{9} \cdot \frac{1}{8} = \frac{1}{10}$ etc.

(or by reasoning):

X has a homogeneous distribution on $\{1,2,..,10\}$,

So $P(X = 2) = \frac{1}{10}$ and $E(X) = \frac{10+1}{2} = 5.5$ (symmetry)

9. **a**. If $X \sim B(25, 0.05)$, then we have: $P(X = 0) = \binom{25}{0} \cdot 0.05^0 \cdot 0.95^{25} = 0.95^{25} \approx 0.277$

b. If X has a Poisson distribution with $\mu = np = 25 \cdot 0.05 = 1.25$,

then $P(X = 0) = \frac{1.25^0 e^{-1.25}}{0!} \approx 0.287$

The probabilities in a. and b. show a difference of (1%), since the Poisson distribution is an approximation of the binomial distribution for "large n and small p", if we choose $\mu = np$). The values of n and p are such that we still can use the binomial tables, which is of course preferable. Of course, the larger n, the smaller the differences between exact binomial probabilities and Poisson approximations will be.

10. **a.** $E(X^k) = \sum_x x^k \cdot P(X = x) = c^k \cdot P(X = c) + 0^k \cdot P(X = 0) = \frac{1}{2}c^k$

b. $E(X) = \frac{1}{2}c$ and $E(X^2) = \frac{1}{2}c^2$

so $var(X) = E(X^2) - (EX)^2 = \frac{1}{2}c^2 - \left(\frac{1}{2}c\right)^2 = \frac{1}{4}c^2$

11. **a**. $\sum_{i=0}^{\infty} c\left(\frac{1}{3}\right)^i = 1$ and $\sum_{i=0}^{\infty}\left(\frac{1}{3}\right)^i = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$, so c must be $\frac{2}{3}$.

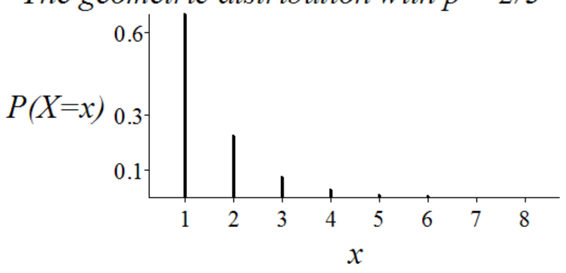
($\sum_{i=0}^{\infty}\left(\frac{1}{3}\right)^i$ is a geometric series $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$

see the appendix "Mathematical Techniques")

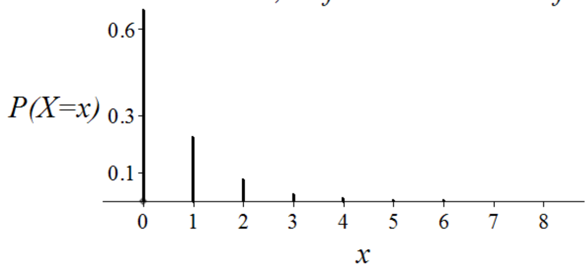
So $P(X = i) = \left(\frac{1}{3}\right)^i \cdot \left(\frac{2}{3}\right)$, $i = 0,1,2, ...$

Compare the distribution of X to the geometric distribution with $p = \frac{2}{3}$ in the accompanying graph.

The geometric distribution with p = 2/3



The same distribution, shifted 1 unit to the left



b. $Y = X + 1$ is geometric ($p = \frac{2}{3}$), so

$E(Y) = \frac{1}{p} = \frac{3}{2}$ and $var(Y) = \frac{1-p}{p^2} = \frac{3}{4}$

c. $E(Y) = E(X + 1) = E(X) + 1 = \frac{3}{2}$,

so $E(X) = \frac{1}{2}$

$var(Y) = var(X + 1) = var(X) = \frac{3}{4}$

12. **a**. All employees could call outside the company with probability $\frac{1}{50}$, independently.

$X =$ "the required number of telephone lines" is $B\left(150, \frac{1}{50}\right)$-distributed.

b. X is approximately Poisson distributed with $\mu = np = 150 \cdot \frac{1}{50} = 3$

(the conditions are fulfilled: $n \geq 25$ and $np < 5$)

The number of lines c must be such that $P(X > c) < 0.05$, so:

$P(X \leq c) > 0.95$. The Poisson-table with $\mu = 3$ reveals that $c = 6$ lines is sufficient.

13. a. The sales trials for the 12 customers can be considered to be 12 Bernoulli trials with success probabilities $p = 0.15$, so $X \sim B(12, 0.15)$.

$E(X) = np = 12 \cdot 0.15 = 1.8$:

$P(X > 1.8) = 1 - P(X = 0) - P(X = 1) = 1 - 0.85^{12} - 12 \cdot 0.85^{11} \cdot 0.15 \approx 0.5565$

b. The demand for incubators is Poisson distributed: the number can be interpreted as the number of rare events on a day (many babies born, only a few will need an incubator).

X is Poisson distributed with parameter $\mu = 2$:

$$P(X > 2) = 1 - P(X = 0) - P(X = 1) - P(X = 2) = 1 - 5e^{-2} \approx 32.33\%$$

c. Here we have draws without replacement from a dichotomous population, so X has a hypergeometric distribution (with parameters $N = 10, R = 3$ and $n = 4$):

$$P(X > 1.2) = 1 - P(X = 0) - P(X = 1) = 1 - \frac{\binom{7}{4}}{\binom{10}{4}} - \frac{3 \cdot \binom{7}{3}}{\binom{10}{4}} = \frac{1}{3}$$

14. a. P("at least one outside class 2") = $1 - P$("all in class 2")

$= 1 - (0.96)^{10} \approx 0.3352$.

b. $X =$ "the number of the 100 eggs outside weight class 2" is $B(100, 0.04)$-distributed, so X is approximately Poisson distributed with $\mu = np = 4$:

$P(X \geq 4) = 1 - P(X \leq 3) = 0.567$ (table, exact with formula: 0.5705)

c. In a. we computed the probability that an arbitrary box of 10 eggs has at least one egg outside class 2: 33.52%. Assuming that this is true for each box independently, then the number of boxes until we have a box with at least one egg outside class 2 is geometrically distributed with $p = 0.3352$.

So the expected number of boxes is $\frac{1}{p} = \frac{1}{0.3352} = 2.984$.

15. a. If the third 6 is the result of the 10th roll with the dice, there are $\binom{9}{2}$ orders with two 6`s and seven non-6`s. The probability of three 6`s and seven non-6`s in a specific order is $\left(\frac{5}{6}\right)^7 \left(\frac{1}{6}\right)^3$. So: $P(\text{third 6 in tenth roll}) = \binom{9}{2}\left(\frac{5}{6}\right)^7 \left(\frac{1}{6}\right)^3 \approx 4.65\%$

b. X = "the number of experiments, necessary to achieve m successes" .

Then: $S_X = \{m, m+1, \ldots.\}$ and $P(X = k) = \binom{k-1}{m-1}(1-p)^{k-m}p^m$, where $k \in S_X$.

Reasoning: there are m successes in k trials, so the probability of one specific order of k successes and $m - k$ failures is $(1-p)^{k-m}p^m$.

The m^{th} success occurs in the k^{th} trial, so the remaining $m - 1$ successes have to be distributed over $k - 1$ positions: $\binom{k-1}{m-1}$ possibilities.

(This distribution is called the **negative binomial distribution**).

16. a. $M = 2$, since $P(X = 1) = \frac{1}{3}$ and $P(X = 2) = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$, so $P(X \leq 2) = \frac{5}{9} > \frac{1}{2}$

b. $M = 2$: see the Poisson table: $P(X \leq 1) < \frac{1}{2}$ and $P(X \leq 2) > \frac{1}{2}$

c. $M = 2$ (as in b.)

d. $M \in [3, 4]$, since $P(X \leq 3) = P(X \geq 4) = \frac{1}{2}$.