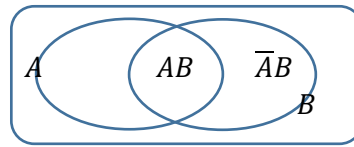


### Solutions exercises chapter 3



1. a.  $P(A|B) = \frac{P(AB)}{P(B)} = \frac{0.30}{0.75} = \frac{2}{5} (= 40\%)$ , since

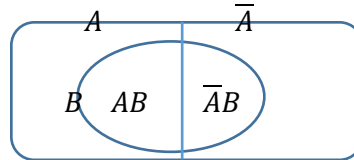
$$P(A \cup B) = P(A) + P(\overline{A}B),$$

$$\text{so } P(\overline{A}B) = P(A \cup B) - P(A) = 0.8 - 0.35 = 0.45$$

$$\text{And } P(B) = P(AB) + P(\overline{A}B) = 0.3 + 0.45 = 0.75$$

b. Use the product rule of independent events (interchange the order of the events)

$$P(ABC) = P(CAB) = P(C)P(A|C)P(B|AC) = \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{10}$$



2.

a. Given probabilities (the events are readily defined):

$$P(A) = 0.98, P(B|A) = 0.97 \text{ en } P(B|\overline{A}) = 0.05$$

It immediately follows that:  $P(\overline{B}|A) = 0.03$ ,  $P(\overline{B}|\overline{A}) = 0.95$  and  $P(\overline{A}) = 1 - P(A) = 0.02$

Requested is  $P(B)$ , which we can compute with the law of total probability:

$$P(B) = P(B|A)P(A) + P(B|\overline{A})P(\overline{A}).$$

$$\text{Substitute the known probabilities: } P(B) = 0.97 \cdot 0.98 + 0.05 \cdot 0.02 = 0.9516$$

b. Requested:  $P(\overline{A}|\overline{B})$ .

$$\text{We will use: } P(\overline{A}|\overline{B}) = \frac{P(\overline{A} \cap \overline{B})}{P(\overline{B})}, P(\overline{A} \cap \overline{B}) = P(\overline{B}|\overline{A})P(\overline{A}), P(\overline{B}) = 1 - P(B)$$

$$P(\overline{B}) = 1 - P(B) = 1 - 0.9516 = 0.0484,$$

$$P(\overline{A} \cap \overline{B}) = P(\overline{B}|\overline{A})P(\overline{A}) = 0.95 \cdot 0.02 = 0.019.$$

$$P(\overline{A}|\overline{B}) = \frac{P(\overline{A} \cap \overline{B})}{P(\overline{B})} = \frac{0.95 \cdot 0.02}{0.0484} = 0.393$$

c. If  $A$  and  $B$  are independent, then  $P(A \cap B) = P(A)P(B)$  or  $P(B|A) = P(B)$  should be true.

We computed  $P(B) = 0.9516$ , which is not the same as  $P(B|A) = 0.97$ .

So  $A$  and  $B$  are **not independent**:  $A$  and  $B$  are dependent.

3. Define the events  $U =$  "Driver is Under influence" and  $B =$  "Blood test is positive"

Then the described probabilities are:  $P(B|U) = 0.75$ ,  $P(B|\overline{U}) = 0.02$  en  $P(U) = 0.05$

Requested is  $P(U|B)$

$$\begin{aligned} \text{Solution: } P(U|B) &= \frac{P(UB)}{P(B)} = \frac{P(B|U)P(U)}{P(B|U)P(U) + P(B|\overline{U})P(\overline{U})} \\ &= \frac{0.75 \cdot 0.05}{0.75 \cdot 0.05 + 0.02 \cdot 0.95} \approx 0.6673 \end{aligned}$$

4. 1. We have a partition of the Sim-cards of mobile phones into the events

$A =$  "Africel",  $G =$  "Gamcel" and  $C =$  "Comium".

Let  $F =$  "Fixed subscription" (the complement of  $F$  is "prepaid").

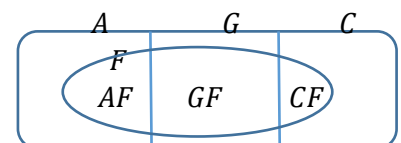
2. Then we know:  $P(F|A) = 0.1$ ,  $P(F|G) = 0.2$  and  $P(F|C) = 0.3$

Furthermore  $P(A) = 2 \cdot P(G)$ .

And finally:  $P(V) = 0.15$

3. Requested is  $P(C)$ .

Solution: Set  $P(G) = x$ , then  $P(A) = 2x$  and  $P(C) = 1 - 3x$



Now apply the law of total probability:

$$P(F) = P(AF) + P(GF) + P(CF)$$

$$= P(F|A)P(A) + P(F|G)P(G) + P(F|C)P(C)$$

So:  $0.15 = 0.1 \times 2x + 0.2 \times x + 0.3 \times (1 - 3x)$

Solving the equation for  $x$ :  $x = P(G) = 0.3$ . So  $P(C) = 1 - 3x = 0.1$

5. 1. Define the events  $L_1 =$  “The drawer with two golden coins is chosen” ,  
 $L_2 =$  “The drawer with two silver coins is chosen” .  
 $L_3 =$  “The drawer with one golden and one silver coins is chosen”, and  
 $A =$  “The first draw results in a golden coin” .

2. Implicitly the following probabilities are given:  $P(L_1) = P(L_2) = P(L_3) = \frac{1}{3}$  and

$$P(A|L_1) = 1, P(A|L_2) = 0 \text{ and } P(A|L_3) = \frac{1}{2}$$

3. Requested is the probability:  $P(L_1|A)$ , since if the first and the second are golden then drawer 1 is opened.

$$P(L_1|A) = \frac{P(AL_1)}{P(A)} \text{ (definition of conditional probability).}$$

$$P(AL_1) \text{ is computed with the product rule: } P(AL_1) = P(L_1)P(A|L_1) = \frac{1}{3} \cdot 1$$

And en  $P(A)$  is computed with the law of total probability:

$$P(A) = P(A|L_1)P(L_1) + P(A|L_2)P(L_2) + P(A|L_3)P(L_3) = \frac{1}{3} \cdot 1 + 0 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{2}$$

$$\text{So } P(L_1|A) = \frac{P(AL_1)}{P(A)} = \frac{1/3}{1/2} = \frac{2}{3}$$

(We can explain this result intuitively: each of the 6 coins in the drawers has the same probability  $\frac{1}{6}$  to be chosen first. If the first one is golden there are 3 equally likely possibilities: two in drawer 1 and one in drawer 3 (one golden en one silver). If it is one of the 2 golden coins in drawer 1, the other one is golden as well. Only in case of the golden coin of drawer 3 the second coin is silver.

So the probability of a second golden one (given that the first golden is) is  $\frac{2}{3}$ .)

6. a. See the sketch of the 5 draws from 5 red and 7 white marbles without replacement.

We can define the event “3 red marbles in 5 draws” =  $R_3$  and  
“face up number 5” =  $F_5$ :

$$P(R_3|F_5) = \frac{\binom{5}{3} \cdot \binom{7}{2}}{\binom{12}{5}} \approx 0.2652$$

	Red	White	Total
Vase	5	7	12
	↓	↓	↓
Draws	3	2	5

- b. If we define  $F_i =$  “Face up number =  $i$ ” and  $R_3 =$  “3 Red” (like in a.), we know that  $P(F_i) = \frac{1}{6}$ ,  $i = 1, \dots, 6$  and the conditional probabilities  $P(R_3|F_i)$  can be determined similarly as  $P(R_3|F_5)$  in a. and  $P(R_3)$  with the law of total probability:

$$P(R_3) = P(R_3F_3) + P(R_3F_4) + P(R_3F_5) + P(R_3F_6)$$

$$= P(R_3|F_3)P(F_3) + P(R_3|F_4)P(F_4) + P(R_3|F_5)P(F_5) + P(R_3|F_6)P(F_5)$$

$$= \sum_{i=3}^6 \frac{\binom{5}{3} \cdot \binom{7}{i-3}}{\binom{12}{i}} \cdot \frac{1}{6} \approx 0.1385$$

7.

- We define the events  $K =$  “student knows the answer” (and we assume the answer is correct in that case) and  $C =$  “correct answer”.
- Given probabilities:  $P(K) = \frac{3}{5}$ ,  $P(C|K) = 1$  and  $P(C|\bar{K}) = \frac{1}{2}$
- Requested:  $P(K|C)$

$$\text{Bayes` Rule: } P(K|C) = \frac{P(KC)}{P(C)} = \frac{P(C|K)P(K)}{P(C|K)P(K) + P(C|\bar{K})P(\bar{K})} = \frac{1 \cdot \frac{3}{5}}{1 \cdot \frac{3}{5} + \frac{1}{2} \cdot \frac{2}{5}} = \frac{3}{4}$$

8. a. We know that  $A$  and  $B$  are independent, so  $P(AB) = P(A) \cdot P(B)$ .

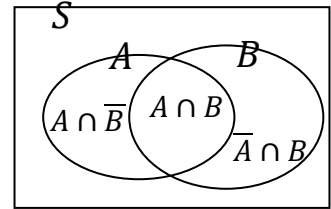
We have to prove that  $P(A\bar{B}) = P(A) \cdot P(\bar{B})$  as well.

The link between  $AB$  and  $A\bar{B}$  is visible in the Venn diagram:

$A = AB \cup A\bar{B}$ , where  $AB$  and  $A\bar{B}$  are mutually exclusive.

$$P(A) = P(AB \cup A\bar{B}) = P(AB) + P(A\bar{B})$$

$$\begin{aligned} \text{So } P(A\bar{B}) &= P(A) - P(AB) = P(A) - P(A)P(B) \\ &= P(A)[1 - P(B)] = P(A) \cdot P(\bar{B}) \end{aligned}$$



Conclusion:  $A$  and  $\bar{B}$  are independent as well, from which it follows that  $\bar{A}$  and  $\bar{B}$  are independent.

- b.  $A, B$  and  $C$  are independent if each pair is independent **and**  $P(ABC) = P(A)P(B)P(C)$

$A$  and  $BC$  are independent as well, because:

$$P(A \cap BC) = P(ABC) = P(A)P(B)P(C) = P(A) \cdot P(BC).$$

9. If  $A$  and  $B$  are independent, then  $P(AB) = P(A) \cdot P(B)$

If  $A$  and  $B$  are mutually exclusive, then  $P(AB) = 0$ .

So mutually exclusive events can only be independent if

$P(AB) = P(A) \cdot P(B) = 0$ , so if  $P(A) = 0$  and/or if  $P(B) = 0$ .

In general mutually exclusive events will be dependent (which is intuitively clear: if  $A$  occurs,  $B$  cannot occur, and reverse. Information about the occurrence of  $A$  is informative for the occurrence of  $B$ ).

10. This is the probability of 6 (independent) trials in a row without a 6:  $\left(\frac{5}{6}\right)^6 \approx 33.49\%$

11.  $X =$  "The number of correct predictions for 12 matches": 12 Bernoulli trials with success probability

$$p = \frac{1}{3}. \text{ Applying the binomial formula: } \sum_{k=10}^{12} P(X = k) = \sum_{k=10}^{12} \binom{12}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{12-k} \approx 0.054\%$$

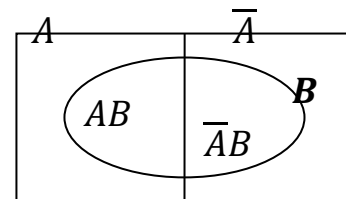
12. a. Requested:  $P(B)$ , where  $B$  is the event that the ELISA test is positive.

We define furthermore  $A$  as the event that a person is really having Antibodies (HIV)

Given probabilities:  $P(B|A) = 0.997$  and  $P(\bar{B}|A) = 0.003$ ,

$$P(B|\bar{A}) = 0.015 \text{ and } P(\bar{B}|\bar{A}) = 0.985,$$

$$P(A) = 0.01 \text{ and } P(\bar{A}) = 0.99$$



Law of total probability:  $P(B) = P(B|A)P(A) + P(B|\bar{A})P(\bar{A})$

$$\text{Substitution: } P(B) = 0.997 \times 0.01 + 0.015 \times 0.99 = 0.02482$$

- b. Requested:  $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)} = \frac{0.997 \times 0.01}{0.02482} \approx 40.2\%$

So: only 40% of the persons with a positive test result is really infected, the others are "false positives".  
(In b. we implicitly applied Bayes' Rule: the denominator  $P(B)$  was computed in a.: without the result in

$$\text{a. we would compute: } P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\bar{A})P(\bar{A})}$$