

# **Slides Probability Theory**

## **Chapter 8**

Ch. 8: Distributions of waiting times

# Discrete: geometric waiting time

If every trial takes a fixed unit of time, then the total (number of unites of) time needed to attain a success has a geometric distribution:

$$P(X = k) = (1 - p)^{k-1} p, \text{ met } k = 1, 2, \dots$$

Properties:

$$E(X) = \frac{1}{p}, \text{ var}(X) = \frac{1-p}{p^2} \text{ and } P(X > k) = (1 - p)^k$$

The **lack of memory property** of the geometric distribution: “the number of trials in the past does not influence the probability of success in future.”

$$P(X > s + t | X > s) = P(X > t)$$

for all  $t$  and  $s$  in  $S_X$

# Continuous waiting times: exponential distribution

$$f_X(x) = \lambda e^{-\lambda x}, \text{ with } x \geq 0 \quad (f_X(x) = 0, x < 0)$$

Properties:

$$E(X) = \frac{1}{\lambda}, \text{ var}(X) = \frac{1}{\lambda^2} \text{ and } P(X > x) = e^{-\lambda x}$$

The exponential distribution has **the lack of memory property**: “the remaining waiting time does not depend on the elapsed waiting time”:

$$P(X > s + t | X > s) = P(X > t) \text{ for all } t \text{ en } s \text{ in } S_X$$

The **geometric** distribution is the **only discrete**, and the **exponential** distribution the **only continuous** distribution that has the lack of memory property.

# Sums of independent waiting times $X_1, \dots, X_n$

if the  $X_i$ 's are exponentially distributed with par.  $\lambda$   
so the total waiting time is  $S_n = \sum_{i=1}^n X_i$

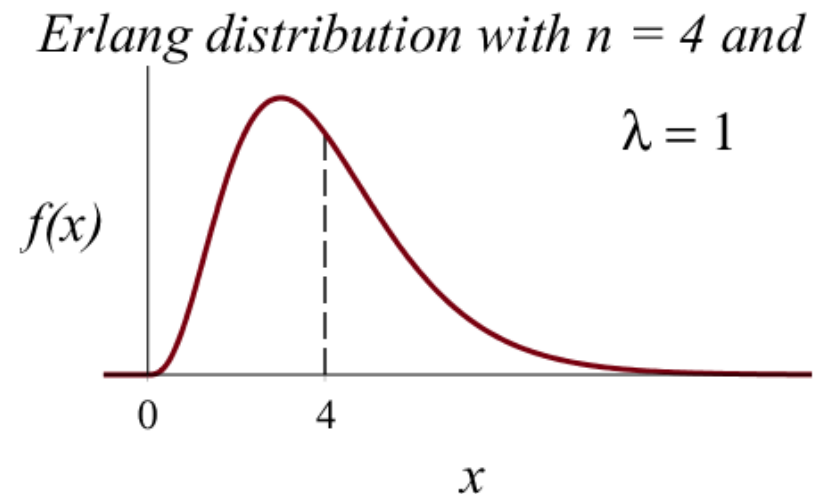
Properties:

1.  $E(S_n) = \frac{n}{\lambda}$  and  $var(S_n) = \frac{n}{\lambda^2}$
2. for large  $n$  ( $> 25$ ) the CLT implies:  $S_n \sim N\left(\frac{n}{\lambda}, \frac{n}{\lambda^2}\right)$

3. 
$$f_{S_n}(s) = \frac{\lambda(\lambda s)^{n-1} e^{-\lambda s}}{(n-1)!}$$

$(s \geq 0)$

**The Erlang distribution**  
with parameters  $n$  and  $\lambda$ .



# Poisson process

for service times / waiting times / interarrival times

If  $X_1, \dots, X_n$  are **independent** and **exponentially distributed** with parameter  $\lambda$ , then we have:

- The total  $S_n = \sum X_i$  has an **Erlang distribution** with **parameters  $n$  and  $\lambda$** .
- The number of arrivals (serviced customers) in the period  $[0, s]$  has **Poisson** distribution with  $\mu = \lambda s$ .

The last property can be proven, by deriving:

$$F_{S_n}(s) = 1 - \sum_{k=0}^{n-1} \frac{(\lambda s)^k e^{-\lambda s}}{k!} \quad (s \geq 0)$$