Slides
Probability Theory
Chapter 3

Ch. 3: Conditional Probability and Independence

Overview Basic Probability in Ch. 1+2

Axioms of Kolmogorov: 1. $P(A) \geq 0$

2. $P(S) = 1$

3. $P(\cup_i A_i) = \sum_i P(A_i)$,

if the A_i`s are mutually exclusive.

Complement rule: $P(\overline{A}) = 1 - P(A)$

Addition Rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

For **disjoint events** we have: $P(A \cap B) = 0$

Symmetric probability space (Laplace): $P(A) = \frac{N(A)}{N(S)}$

number of permutations of n out of N is $\frac{N!}{(N-n)!}$

number of combinations of n out of N is $\binom{N}{n} = \frac{N!}{n!(N-n)!}$

Conditional probability

"Probabilities within a part *A* of the sample space *S*"

Definition: $\boldsymbol{P(B|A) = \dfrac{P(A \cap B)}{P(A)}}$,

$if\ P(A) > 0$

From this definition the **Product rule** follows**:**

$$\boldsymbol{P(B \cap A) = P(B|A)P(A)}$$

or: $\boldsymbol{P(A \cap B) = P(A|B)P(B)}$

"Conditioning w.r.t. *A* and *B, respectively*"

Product rule for n events:

$$\boldsymbol{P(A_1 A_2 \ldots A_n) = P(A_1)P(A_2|A_1) \ldots P(A_n|A_1 \ldots A_{n-1})}$$

(S, $P(.\,|A)$ **)** is a probability space (Kolmogorov)

An application of Bayes' Rule

Success proportions at the Dutch HBO-colleges

Previous education	**MBO**	**HAVO**	**VWO**
Proportion students	**20%**	**50%**	**30%**
Success proportion	**80%**	**60%**	**90%**

Answer the following questions:

1. What is the overall success proportion at the HBO?

2. Determine the probability that a HBO-graduate had his previous training at VWO

Solution

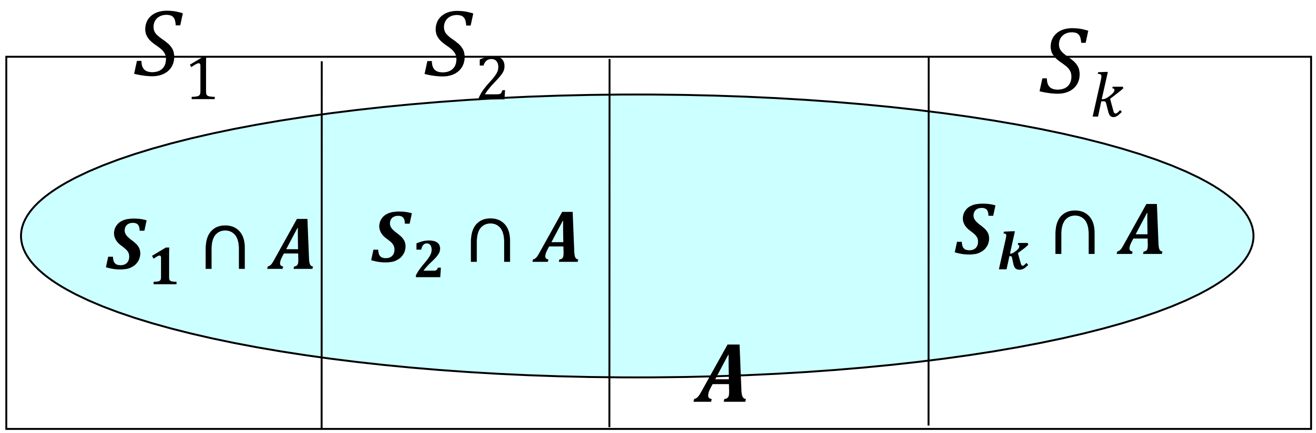
- **Define:** $\boldsymbol{A}$ = "student succeeds"
  S_1 = "previous training is MBO"
  S_2 and S_3 similar for HAVO and VWO resp.
- **Probabilities:** $P(S_1)=0.20$, $P(S_2)= 0.50$, $P(S_3)=0.30$,
  $P(A|S_1) = 0.80$, $P(A|S_2) = 0.60$ and $P(A|S_3) = 0.90$
- **Apply the rules** of total probability and Bayes:

1. $$\boldsymbol{P(A) = P(A \cap S_1) + P(A \cap S_2) + P(A \cap S_3)}$$
   $$\boldsymbol{= P(A|S_1)P(S_1) + P(A|S_2)P(S_2) + P(A|S_3)P(S_3)}$$
   $$= 0.80\times0.20 + 0.60\times0.50 + 0.90\times30 = \underline{\mathbf{0.73}}$$

2. $$\boldsymbol{P(S_3|A) = \frac{P(S_3 \cap A)}{P(A)} = \frac{P(A|S_3)P(S_3)}{P(A)} = \frac{0.9\times0.3}{0.73} \approx 0.37}$$

Rules of total probability and Bayes

Given:



- $P(S_i)$ and $P(A|S_i)$
- Then:

$$P(A) = P(A \cap S_1) + \cdots + P(A \cap S_k)$$
$$= P(A|S_1)P(S_1) + \cdots + P(A|Sk)P(Sk)$$

Law of total probability

And:

$$P(S_1|A) = \frac{P(S_1 \cap A)}{P(A)}$$

Bayes` rule

$$= \frac{P(A|S_1)P(S_1)}{P(A|S_1)P(S_1) + \cdots + P(A|S_k)P(S_k)}$$

Independence of events A and B:

"B does not give information about (the probability of) A"

$$P(A|B) = P(A) \Leftrightarrow \frac{P(A \cap B)}{P(B)} = P(A) \Leftrightarrow P(B|A) = P(B)$$

Definition: A and B are **independent,**

if $P(A \cap B) = P(A) \cdot P(B)$

A, B and C are independent if $P(A \cap B) = P(A) \cdot P(B)$

$P(A \cap C) = P(A) \cdot P(C)$

$P(B \cap C) = P(B) \cdot P(C)$

and $P(A \cap B \cap C) = P(A) \cdot PB) \cdot P(C)$

$A_1, \ldots, A_n$ **are independent** if the equality holds for any subsequence $A_{i_1}, \ldots, A_{i_k}$ $(k = 2, .., n)$

Independent experiments

Experiments are said to be independent if events related to different experiments are independent.

Usually experiments are **assumed** to be independent.

E.g. **Bernoulli experiments** or **Bernoulli trials:**

1. A sequence of **independent** experiments. (Repetitions of the same experiment)
2. **Two possible outcomes**: “Success” and “Failure”
3. Constant “Success”-probability p.

Application 1 (**geometric** formula)

X = the required number of Bernoulli trials until the first success occurs.

$$\boldsymbol{P(X=k)=(1-p)^{k-1}p} \text{ , where } \boldsymbol{k = 1, 2, \ldots}$$

Application 2 (**binomial** formula)

X = the number of successes in n Bernoulli trials

$$\boldsymbol{P(X=k)=\binom{n}{k}p^k(1-p)^k, k=0,1,..,n}$$

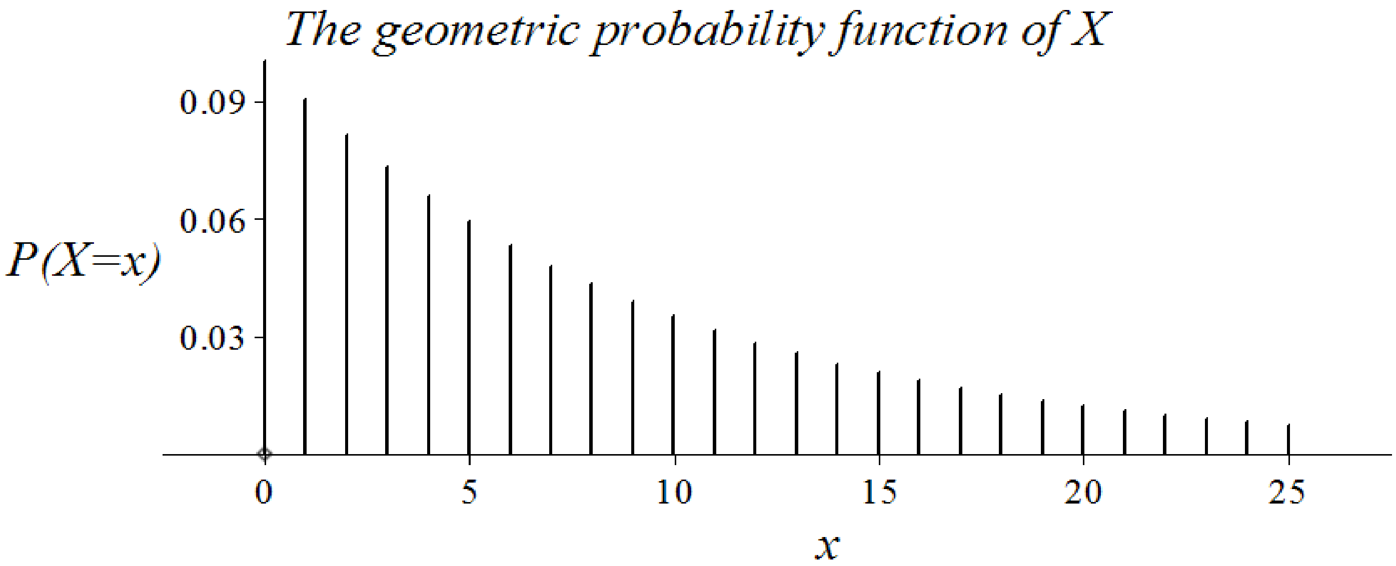
The numerical variables X are called **random variables** → chapter 4 .

Example geometric distribution

Suppose that 10% of the passing cars are Mercedes.

$X =$ "the number van the first passing Mercedes"

$$\boldsymbol{P(X = k) = 0.9^{k-1} \cdot 0.1}\text{, with } \mathbf{k = 1, 2, 3, \ldots}$$



We can reason that on the expected value is

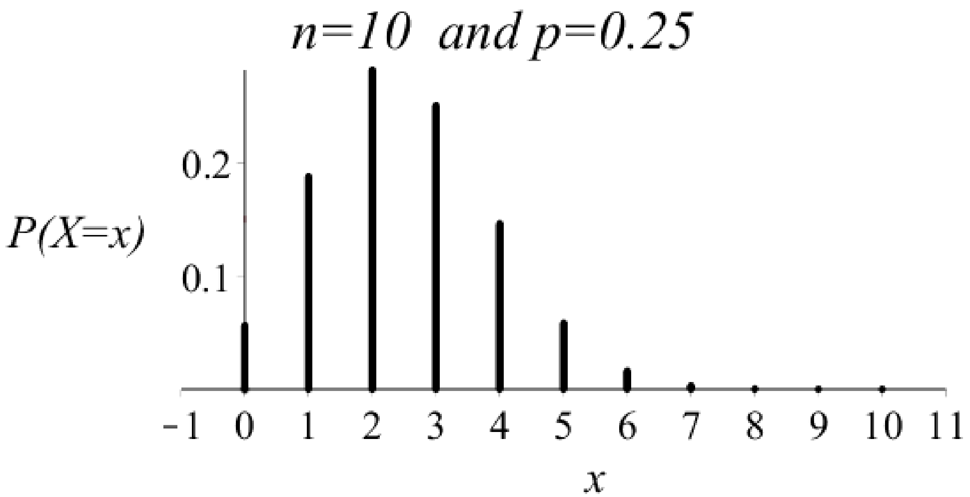
$$\boldsymbol{E(X)} = \frac{1}{0.1} = \mathbf{10} \text{ and } P(X > 10) = 0.9^{10} \approx 34.9\%$$

The binomial distribution – 2 examples

Applicable if we have $\boldsymbol{n}$ (ind.) Bernoulli trials with success rate $\boldsymbol{p}$

Example: X = the number of correct random answers to 10 MC-items:
here $\boldsymbol{n} = \mathbf{10}$ and $\boldsymbol{p} = ¼$

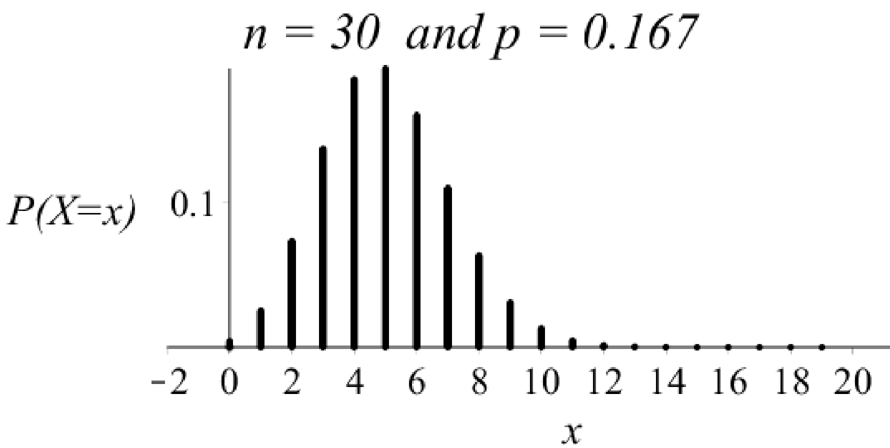
$$P(X = x) = \binom{10}{x}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{10-x},\ x = 0,\ \ldots,10$$



Example:
X = The number of sixes in 30 rolls of a dice.
So $\boldsymbol{n} = \mathbf{30}$ and $\boldsymbol{p} = \frac{1}{6}$

$$P(X = x) = \binom{30}{x}\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{30-x},$$

$$x = 0,\ \ldots,\ 30$$



What is the expected number of correct answers and resp. sixes?

Summary Chapter 3

- **Conditional Probability:** $P(A|B) = \frac{P(A \cap B)}{P(B)}$
- **General product rule:** $P(A \cap B) = P(A|B)P(B)$
- **Product rule in case of independence:**

$$P(A \cap B) = P(A)P(B)$$

- **Law of Total Probability:**

$$P(A) = P(A|S_1)P(S_1) + \cdots + P(A|S_n)P(S_n)$$

- **Bayes` rule:**

$$P(S_1|A) = \frac{P(S_1 \cap A)}{P(A)} = \frac{P(A|S_1)P(S_1)}{P(A|S_1)P(S_1) + \cdots + P(A|S_k)P(S_k)}$$