Slides
Probability Theory
Chapter 2

Combinatorial probability

Computing probabilities combinatorically is possible when all outcomes are **equally likely: count the numbers.**

- The "**art of counting**" can be often be if we have a random sample taken from a finite population at hand: if we arbitrarily choose one element of the population, then evry outcome has the **same probability of occurrence**.
- **Example:**
  A car company buys a group of 20 old VW Golf cars, of which the company does not know that 5 suffer from severe hidden defects.
  The company decides to check a 4 of the 20 cars at random.

  What is the probability that:

1. 2 of the 4 checked cars suffer from severe defects?
2. None of the 4 cars suffer from severe defects?

Basic rules of combinatorics

The product rule: if partial experiment 1 has m outcomes and partial experiment 2 n outcomes (no matter what the result of experiment 1 was), then the combination of the two experiments has $m \times n$ outcomes.

“Drawing n **individuals from a population of** N **individuals** can be executed in different ways:

- Drawing with or without replacement:
  With replacement → repeated outcomes possible
- Two kinds of outcomes:
  permutations: **ordered outcomes**
  combinations: unordered outcomes,
  subsets of the population

In statistics a **random sample of** n **out of** N usually consists of **combinations** **<u>without</u>** **replacement.**

Examples of counting

1. **We roll a dice 3 times:**
   3 partial experiments with each 6 equally likely outcomes:
   Every (ordered) outcome, such as (1,5,1) has probability $\frac{1}{216}$,
   since there are $\mathbf{6 \times 6 \times 6 = 216}$ outcomes.
2. **How many lists of 6 participants of a song contest are possible, if the order of appearance is randomly chosen?**
   Determine the number of permutations- without replacement:
   $\mathbf{6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720}$**.** **Notation: 6! ("6 factorial")**
3. Generalization**:** the number of lists with $\boldsymbol{n}$ **participants** is $\boldsymbol{n}!$
4. **How many lists of 6 persons are there if we choose them arbitrarily from a group of 15 persons** (without replacement)?
   15 outcomes for the 1st choice, 14 for the 2nd, etc.:

$$15 \times 14 \times 13 \times 12 \times 11 \times 10 = \frac{15!}{10!}$$

Numbers of **permutations and combinations of n out of N**

- The **numbers of permutations** (variations) of **n out of N**

$$N \times (N-1) \times \cdots \times (N-n+1) = \frac{N!}{(N-n)!}$$

Calculator: use the button *n*P*r*

- The **numbers of combinations of 6 out of 15**
  (= the numbers **subsets of 6** elements chosen out of 15):

  For **each** combination of 6 there are 6! permutations.
  So: the number of permutations = (# of combinations) × 6!

  The number of combinations $= \frac{\frac{15!}{9!}}{6!} = \frac{15!}{6!9!}$

- In general: the **number of combinations of *n* out of *N*** is

$$\frac{(\#permutations\ of\ n\ out\ of\ N)}{n!} = \frac{N!}{n!(N-n)!}$$ **(say: "*N* choose *n*")**

Notation: *binomial coefficient* $\binom{N}{n}$, calculator button: ***n*C*r***

Combinatorics and probabilities of sample results

If a **combination** of $\boldsymbol{n}$ person is chosen arbitrarily and without replacement from a population of N persons, then the probability of each combination of n persons is the same: $\dfrac{\mathbf{1}}{\binom{\boldsymbol{N}}{\boldsymbol{n}}}$

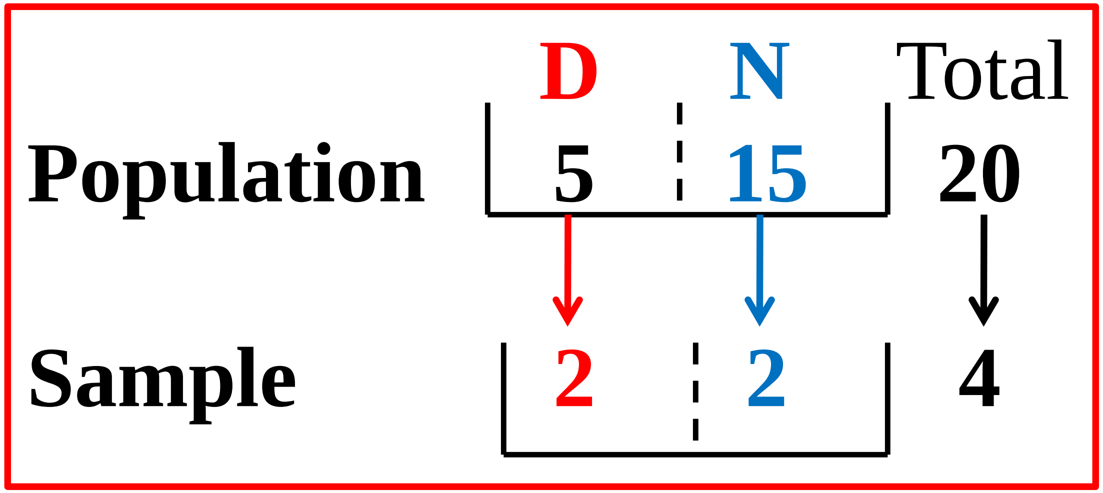
- **Solution of the defect Golf cars problem**

1. For a combination of 2 **D**efect and 2 **N**on-defect cars we have to choose (first) 2 out of 5 defect Golfs and then 2 out of 15 non-defect Golfs. # of these combinations is $\binom{5}{2} \times \binom{15}{2}$ on a total of $\binom{20}{4}$.

$$P(2\boldsymbol{D} \text{ and } 2\ \boldsymbol{N}) = \frac{\binom{5}{2} \times \binom{15}{2}}{\binom{20}{4}}$$

2. $P(\text{no Defect Golfs})$

$$= \frac{\binom{5}{2} \times \binom{15}{2}}{\binom{20}{4}} \approx 0.217$$



Combinatorial Probability

Applying Laplace's definition: $P(A) = \frac{N(A)}{N(S)}$

For k **draws out of** n (different) "**things**", we distinguish:

		Way of drawing	
		Without replacement	**With replacement**
Type of out-come	**Ordered**	$\frac{n!}{(n-k)!}$ ***permutations***	n^k **permutations with repetition**
	Un-ordered	$\binom{n}{k}$ ***combinations***	***Combinations with repetition:*** **non-symmetric!**

Generalizing the example: **hypergeometric formula**

Applicable for situation **of draws without replacement and unordered outcomes, e.g choose n out of N** balls:
***R* red** and $N - R$ **white.**

	red	white	total
population:	R	$N - R$	N
	↓	↓	↓
sample	k	$n - k$	n

The diagram shows the outcome of k red and $n - k$ white.

Random variable
X = "the number of red balls after n draws"

$$P(X = k) = \frac{\binom{R}{k}\binom{N-R}{n-k}}{\binom{N}{n}},$$

$$k = 0, \ldots, n$$

X has a **hypergeometric distribution.**

Instruments of Combinatorial Probability

- **Product rule**: the numbers of outcomes of partial outcomes can be multiplied (if "independent")
- Use in case of "arbitrary draws from populations"the so called **vase model** to visualize the problem.
- Are the draws **with** or **without replacement**? Can outcomes repeatedly occur?
- Is the order of the outcomes of interest for the problem? **Permutations** (orders, variations) or **Combinations** (unordered, subsets).
- Is **the probability of the complement** easier to compute?
- Avoid "**double counting"** of outcomes.

Overview Basic Probability in Ch. 1+2

Axioms of Kolmogorov: 1. $P(A) \geq 0$

2. $P(S) = 1$

3. $P(\cup_i A_i) = \sum_i P(A_i)$,

if the A_i`s are mutually exclusive

Complement rule: $P(\overline{A}) = 1 - P(A)$

Addition Rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

For **disjoint** events we have: $P(A \cap B) = 0$

Symmetric probability space (Laplace): $P(A) = \frac{N(A)}{N(S)}$

number of permutations of n out of N is $\frac{N!}{(N-n)!}$

number of combinations of n out of N is $\binom{N}{n} = \frac{N!}{n!(N-n)!}$

Examples of draws without replacement

How many matches are there in a competition of 6 teams?

- A full competition (home and away games a-b and b-a)?
  Solution: $6 \cdot 5$ games: $\frac{6!}{4!}$ permutations of 2 out of 6.
- A tournament (unordered pairs $\{a, b\}$ of 2 teams)
  Sol.: $\frac{6 \cdot 5}{2}$ games: $\frac{6!}{4!2!} = \binom{6}{2}$ combinations of 2 out of 6.

How many (different) compositions are possible, if you have 10 persons to make

- A board with the chair, secretary and treasurer?
  Solution: $10 \cdot 9 \cdot 8 = \frac{10!}{(10-3)!}$ **permutations of 3 out of 10**
- A **committee** of 3 persons?
  $\frac{10 \cdot 9 \cdot 8}{3!} = \frac{10!}{3!7!} = \binom{10}{3}$ **combinations of 3 out of 10**