Lecture 6 – Second order differential equations

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Second order differential equations

Example

Solve the differential equation

$$y'' + 5y' + 4y = 0. \tag{1}$$

by substituting $y = e^{rx}$

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$$\begin{aligned} y'' + 5y' + 4y &= r^2 e^{rx} + 5r\,e^{rx} + 4e^{rx} \\ &= \big(r^2 + 5r + 4\big)e^{rx} = 0 \end{aligned}$$

  if and only if

$$r^2 + 5r + 4 = 0. \tag{2}$$

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- This equation (2) has two roots: $r_1 = -1$ and $r_2 = -4$.
- The solutions of (1) are
$$e^{-x} \text{ and } e^{-4x}.$$

More solutions

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  For all $c_1, c_2 \in \mathbb{R}$, the functions $c_1 e^{-x}$ and $c_2 e^{-4x}$ are solutions as well.

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  For all $c_1, c_2 \in \mathbb{R}$, the function $c_1 e^{-x} + c_2 e^{-4x}$ is a solution as well.

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- All solutions of $y'' + 5y' + 4y = 0$ are given by

$$y(x) = c_1 e^{-x} + c_2 e^{-4x}, \qquad c_1, c_2 \in \mathbb{R}$$

The characteristic equation

Definition

Consider the differential equation

$$ay'' + by' + cy = 0. \tag{1}$$

The equation

$$a\,r^2 + b\,r + c = 0$$

is called the characteristic equation of (1).

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$$a\,r^2 + b\,r + c = 0$$

is called the **characteristic equation of (1)**.

- If r is a root of the characteristic equation, then $y(x) = e^{rx}$ is a solution of (1).

The general solution

Theorem

Consider the second order differential equation

$$ay'' + by' + cy = 0$$

with a, b*, and* c *constant.*

If $b^2 - 4ac > 0$*, then the solutions are given by*

$$y(x) = \alpha e^{r_1 x} + \beta e^{r_2 x}, \qquad \alpha, \beta \in \mathbb{R}. \tag{1}$$

with

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

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- The differential equation

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Problem 1 The differential equation

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has characteristic equation $r^2 + 2r + 1 = 0$, which has discriminant 0.

Problem 2 The differential equation

$$y'' + 2y' + 10y = 0.$$

has characteristic equation $r^2 + 2r + 10 = 0$, which has negative discriminant.

Characteristic equation with zero discriminant

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$$ay'' + by' + cy = 0$$

with a, b, and c constant. If $b^2 - 4ac = 0$, then the solutions are given by

$$y(x) = \alpha e^{rx} + \beta x e^{rx}, \qquad \alpha, \beta \in \mathbb{R}.$$

with

$$r = \frac{-b}{2a}.$$

- If the discriminant is non-zero, the function $x\, e^{rx}$ will not be a solution.

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Find the general solution of the differential equation

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- Since $b^2 - 4ac = 2^2 - 4 \cdot 1 \cdot 1 = 0$, the solution is given by

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- If you try $y(x) = x\, e^{rx}$, then you will also find $r = -1$ (Please check).

Characteristic equation with negative discriminant

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Solve the differential equation

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- Try the exponential $e^{\lambda x}$:

$$\lambda^2 e^{\lambda x} + 2\lambda e^{\lambda x} + 10 e^{\lambda x} = 0$$

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- From this follows

$$\lambda^2 + 2\lambda + 10 = 0 \qquad \Longleftrightarrow \qquad \lambda_{1,2} = -1 \pm 3i,$$

Characteristic equation with negative discriminant

- The exponential solutions of

$$y'' + 2y' + 10y = 0.$$

are given by

$$y(x) = e^{(-1+3i)x} \text{ and } y(x) = e^{(-1-3i)x}$$

Characteristic equation with negative discriminant

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- Thus the general solution is

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with arbitrary $\alpha, \beta \in \mathbb{C}$.

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- Question: can we write these in a real form?

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- The solution of

$$y'' + 2y' + 10y = 0. \tag{1}$$

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- By restricting yourselves to real values for c_1 and c_2, you get all **real** solutions of (1).

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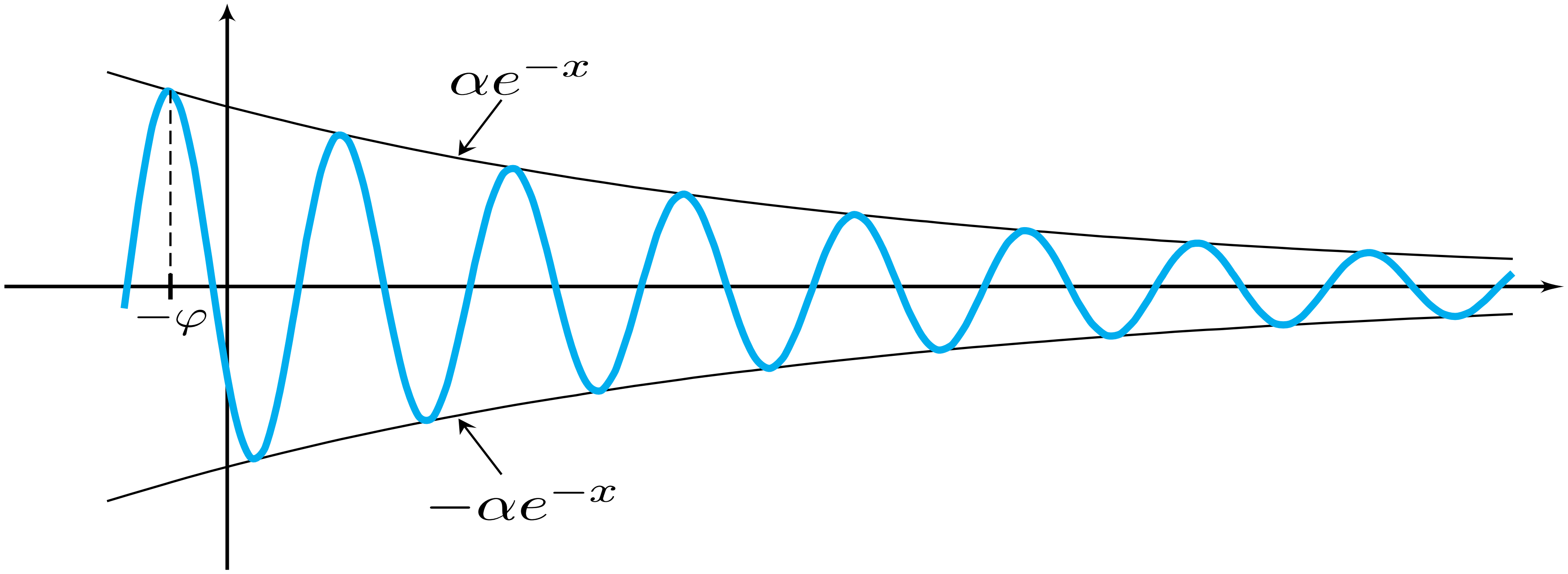
- By restricting yourselves to real values for c_1 and c_2, you get all **real** solutions of (1).
- The value of the constants c_1 and c_2 are inferred from the initial conditions.

Second real form of the answer

There is another way of writing this answer:

$$c_1 e^{-x}\cos(3x) + c_2 e^{-x}\sin(3x) = \alpha e^{-x}\cos(3x+\varphi),$$

for some constants $\alpha \geq 0$ and $\varphi \in [0, \pi)$.



Characteristic equation with non-zero discriminant

Theorem

- *Consider the second order differential equation*

$$ay'' + by' + cy = 0$$

with a, b*, and* c *constant. If* $b^2 - 4ac \neq 0$*, then the solutions are given by*

$$y(x) = \alpha e^{\lambda_1 x} + \beta e^{\lambda_2 x}, \qquad \alpha, \beta \in \mathbb{C}.$$

with

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- *If* $b^2 - 4ac < 0$*, then the* **real** *solutions can be written as*

$$y(x) = e^{\sigma x}\left[c_1 \cos(\omega x) + c_2 \sin(\omega x)\right], \qquad c_1, c_2 \in \mathbb{R}.$$

with $\sigma = \mathrm{Re}(\lambda_1)$ *and* $\omega = \mathrm{Im}(\lambda_1)$*.*

Seismic damper for the Sutong bridge (China), © 2016 Taylor Devices

Inhomogeneous equations

Problem

Find the solutions of

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- Notice that $y_p'(x) = y_p''(x) = 0$, hence substituting y_p in the equation gives
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- Notice that $y_p'(x) = y_p''(x) = 0$, hence substituting y_p in the equation gives
$$y_p(x) = 2.$$
- This is only one solution. How do we get more solutions?

Inhomogeneous equations

Consider the differential equation

$$y'' + 2\,y' + 10\,y = 20. \tag{1}$$

- We know that $y_p(x) = 2$ is a particular solution. Define

$$\tilde{y} = y - 2$$

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$$\tilde{y}(x) = e^{-x}\left[c_1 \cos(3x) + c_2 \sin(3x)\right].$$

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$$\tilde{y}(x) = e^{-x}\left[c_1 \cos(3x) + c_2 \sin(3x)\right].$$

- Hence all solutions of (1) are given by

$$y(x) = c_1 e^{-x} \cos(3x) + c_2 e^{-x} \sin(3x) + 2.$$

Inhomogeneous equations

Theorem

Consider the second order differential equation

$$ay'' + by' + cy = f, \qquad (1)$$

with a, b, *and* c *constant and* f *a function.*
The solution y *is given as*

$$y = y_h + y_p,$$

where y_h *is the solution to the homogeneous equation*

$$ay'' + by' + cy = 0, \qquad (2)$$

and y_p *is a particular solution of* (1).

Particular solutions

To find a particular solution the following table provides candidates

f	candidate y_p
d (constant)	α (constant)
$cx + d$	$\alpha x + \beta$
$e^{\mu x}$	$\alpha e^{\mu x}$
$\cos(\omega x)$ or $\sin(\omega x)$	$\alpha \cos(\omega x) + \beta \sin(\omega x)$

Initial conditions

Problem

Find the solutions of

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- Use the initial conditions to find c_1 and c_2:

$$\begin{aligned} y'(x) = c_1 &\left[-e^{-x}\cos(3x) - 3e^{-x}\sin(3x)\right] + \\ c_2 &\left[-e^{-x}\sin(3x) + 3e^{-x}\cos(3x)\right]. \end{aligned}$$

Initial conditions

- We have

$$
\begin{aligned}
y(x) &= c_1 e^{-x}\cos(3x) + c_2 e^{-x}\sin(3x) + 2 \\
y'(x) &= c_1\left[-e^{-x}\cos(3x) - 3e^{-x}\sin(3x)\right] + \\
&\quad c_2\left[-e^{-x}\sin(3x) + 3e^{-x}\cos(3x)\right] \\
y(0) &= 0 \\
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$$0 = y(0) = c_1 e^{-0}\cos(0) + c_2 e^{-0}\sin(0) + 2$$

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- So

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Initial conditions

- We have

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- So

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- Therefore $c_1 = -2$, $c_2 = 4$, and

$$\boxed{y(x) = -2e^{-x}\cos(3x) + 4e^{-x}\sin(3x) + 2}$$

Hyperbolic sine and cosine

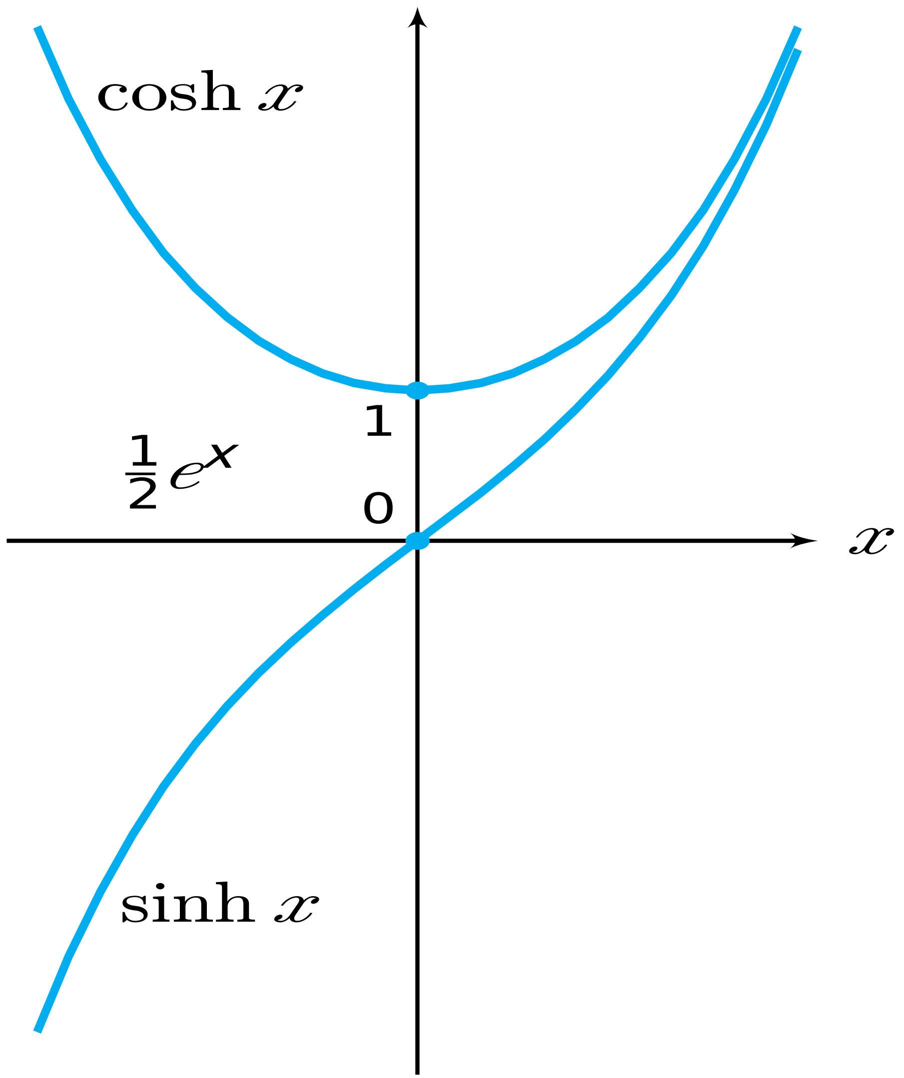
Definition

- The hyperbolic cosine is the function

$$\cosh x = \frac{e^x + e^{-x}}{2}$$ ▷

- The hyperbolic sine is the function

$$\sinh x = \frac{e^x - e^{-x}}{2}$$ ▷



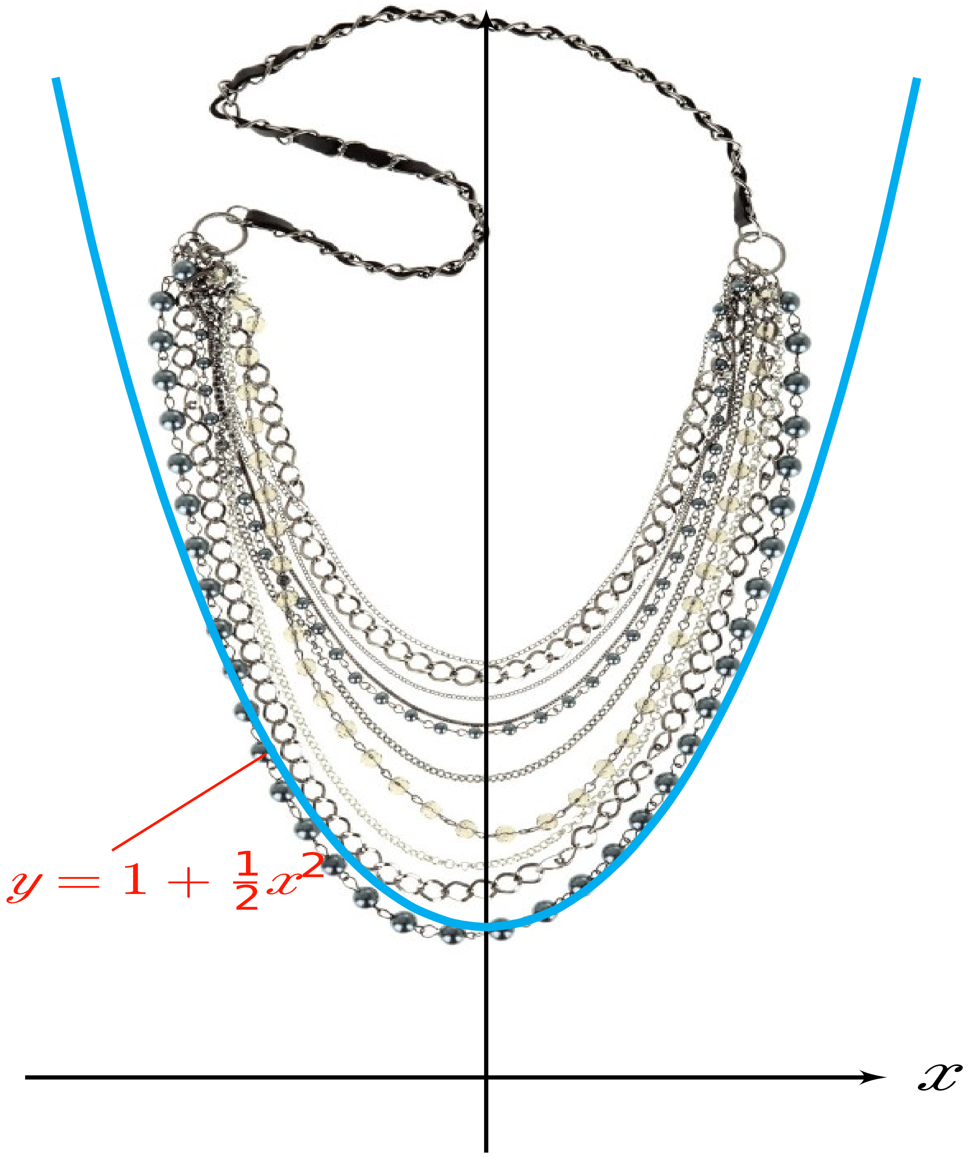
The catenary

- The hyperbolic cosine can be used to describe the catenary, the shape of a chain suspended between two points.

The catenary

- The hyperbolic cosine can be used to describe the catenary, the shape of a chain suspended between two points.
- It looks like a parabola, but it isn't!

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$



Why 'hyperbolic'?

- Observe that

$$\cosh^2 x - \sinh^2 x = \frac{\left(e^x + e^{-x}\right)^2}{4} - \frac{\left(e^x - e^{-x}\right)^2}{4}$$

$$= \tfrac{1}{4}\Big[\left(e^{2x} + 2e^x e^{-x} + e^{-2x}\right) - \left(e^{2x} - 2e^x e^{-x} + e^{-2x}\right)\Big]$$

Why 'hyperbolic'?

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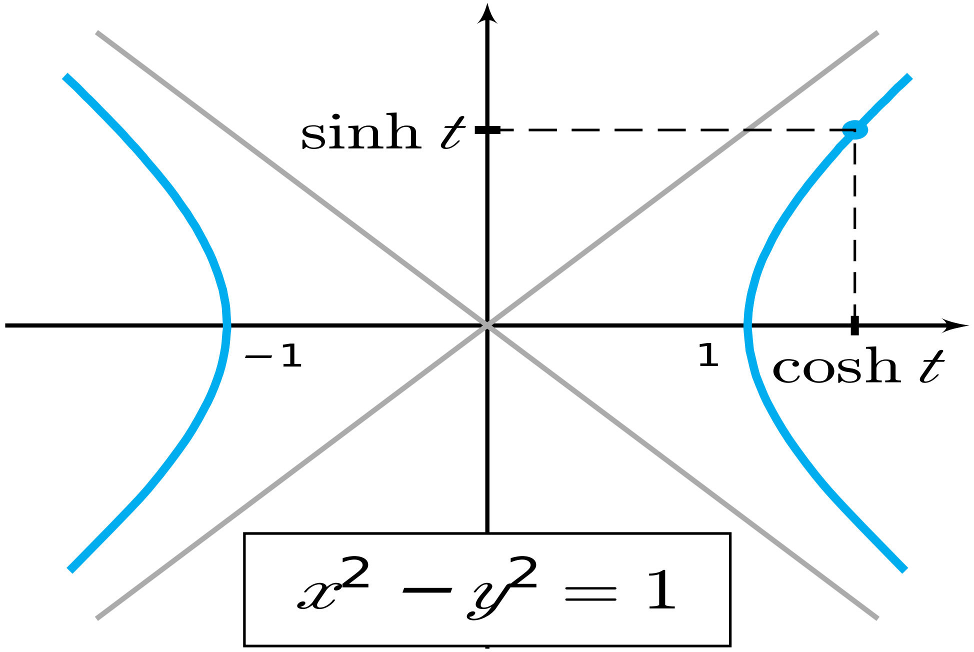
$$= e^x e^{-x} = 1 \triangleright$$

- For all x we have

$$\boxed{\cosh^2 x - \sinh^2 x = 1}$$



Why '(co)sine'?

- The hyperbolic sine and cosine have properties that are very similar to the properties of $\sin$ and $\cos$:

(1) $\cosh(-x) = \cosh x$

(2) $\sinh(-x) = -\sinh x$

(3) $\cosh^2 x - \sinh^2 x = 1$

(4) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

(5) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

(6) $\cosh(2x) = \cosh^2 x + \sinh^2 x$

(7) $\sinh(2x) = 2 \sinh x \cosh x$

(8) $\frac{d}{dx} \sinh x = \cosh x$

(9) $\frac{d}{dx} \cosh x = \sinh x$

Hyperbolic functions and complex numbers

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$$e^z = e^x e^{iy} = e^x(\cos y + i \sin y).$$

Hyperbolic functions and complex numbers

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for arbitrary $z \in \mathbb{C}$.

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  for arbitrary $z \in \mathbb{C}$.

- With this definition we have for all real x:

$$\boxed{\cos x = \cosh(ix)} \quad \text{and} \quad \boxed{\sin x = -i \sinh(ix)}$$

A useful property

- From $\cosh^2 x - \sinh^2 x = 1$ follows

$$\frac{d}{dx} \sinh x = \cosh x = \sqrt{1 + \sinh^2 x},$$

in other words, $\sinh x$ is a solution of the differential equation

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in other words, $\sinh x$ is a solution of the differential equation

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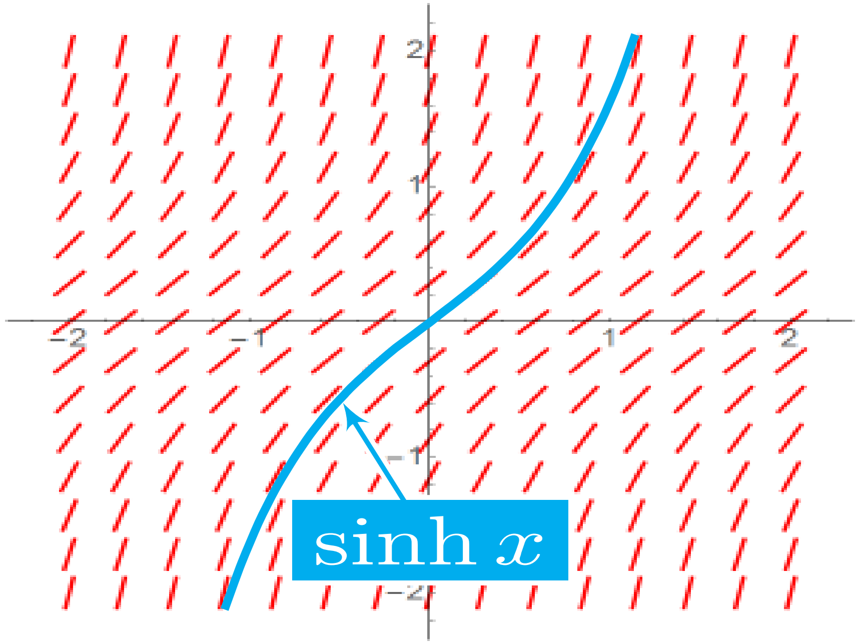
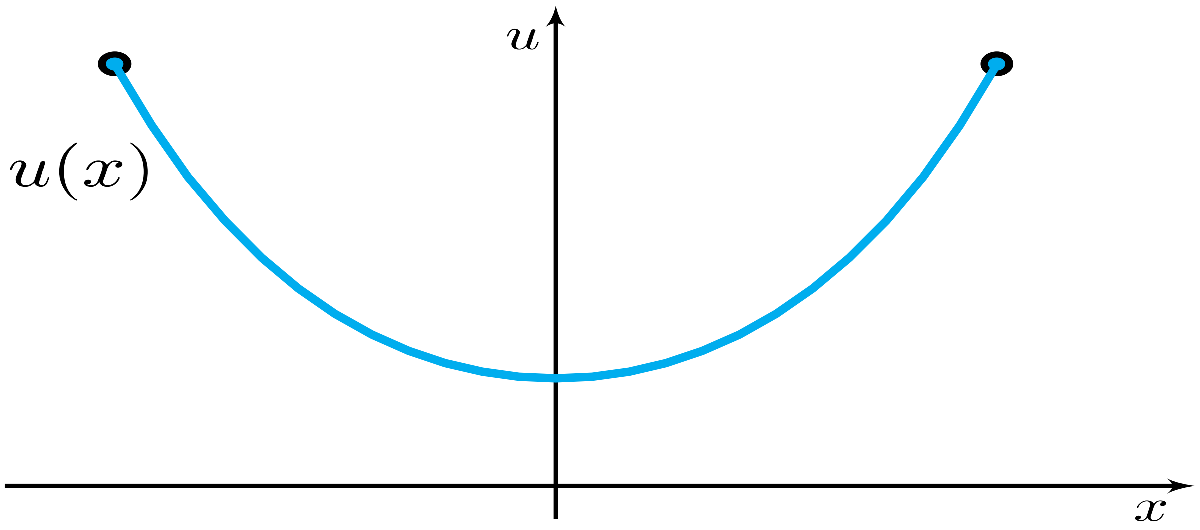
- The function $\sinh x$ is the unique solution of the initial value problem

$$\begin{cases} y' = \sqrt{1+y^2}, \\ y(0) = 0. \end{cases}$$



The catenary



- A chain is suspended between two fixed points.

$$u(x) = \cosh(x)$$

Summarizing Exercise

Find the real-valued function *y* which solves the second order
differer

$$y'' - y' - 2y = -\cos(x) - 3\sin(x)$$

such that

$$y(0) = 1 \quad ; \quad y'(0) = 3$$

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