Lecture 5 – Complex numbers

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Mathematics B2: Newton

-Contents-

- ❑√ Integrals
- ❑√ Calculation techniques for integrals
- √ ❑ Power and Taylor series
- ❑√ First order ODEs
- ❑ Complex numbers
- ❑ Second order ODEs

Topics

- Introduction to second order differential equations.
- Complex numbers
  - Calculations with complex numbers
  - Eulers formula
  - Solving polynomial equations like $z^4 = 16$.

First order differential equations

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Consider the first order differential equation

$$5y' + 4y = 0.$$

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- We find the solution $\boxed{y(x) = e^{-\frac{4}{5}x}}$

Trying exponentials in second order differential equations

Substitute $y = e^{rx}$ in

$$y'' + 5y' + 4y = 0.$$

- $(e^{rx})' = re^{rx}, \qquad (e^{rx})'' = r^2e^{rx},$

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- Two solutions of $y'' + 5y' + 4y = 0$ are

$$\boxed{y(x) = e^{-x}} \quad \text{and} \quad \boxed{y(x) = e^{-4x}}$$

Trying exponentials, second example.

Example

Consider next the second order differential equation

$$y'' + 2y' + 10y = 0.$$

- Substituting $y(x) = e^{rx}$ gives

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- However, because $b^2 - 4ac = 2^2 - 4 \cdot 10 < 0$, this equation does not have a real solution. Thus this technique seems not to work.
- We need complex numbers.

Number systems

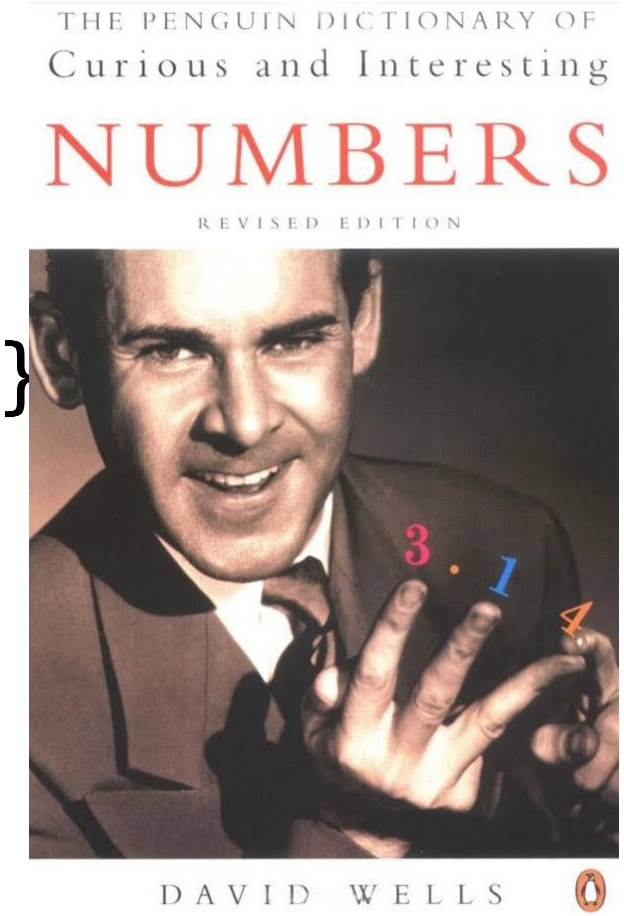
$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$

natural numbers {1, 2, 3, 4, ...}

integers {. . . -3, -2, -1, 0, 1, 2, 3 . . . .}

rational numbers ....

real numbers



Complex numbers

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- or

$$\text{“ } i = \sqrt{-1} \text{ ”}$$

Complex numbers

Definition

A complex number is an expression of the form

$$a + bi,$$

where a and b are real numbers.

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- Sometimes we write $a + ib$ instead of $a + bi$.
- If $z = a + bi$, then a is called the real part of z and b is called the imaginary part of z.
- We denote $a = \operatorname{Re} z$ and $b = \operatorname{Im} z$.

Complex numbers

We can do calculations with complex numbers in the usual way.

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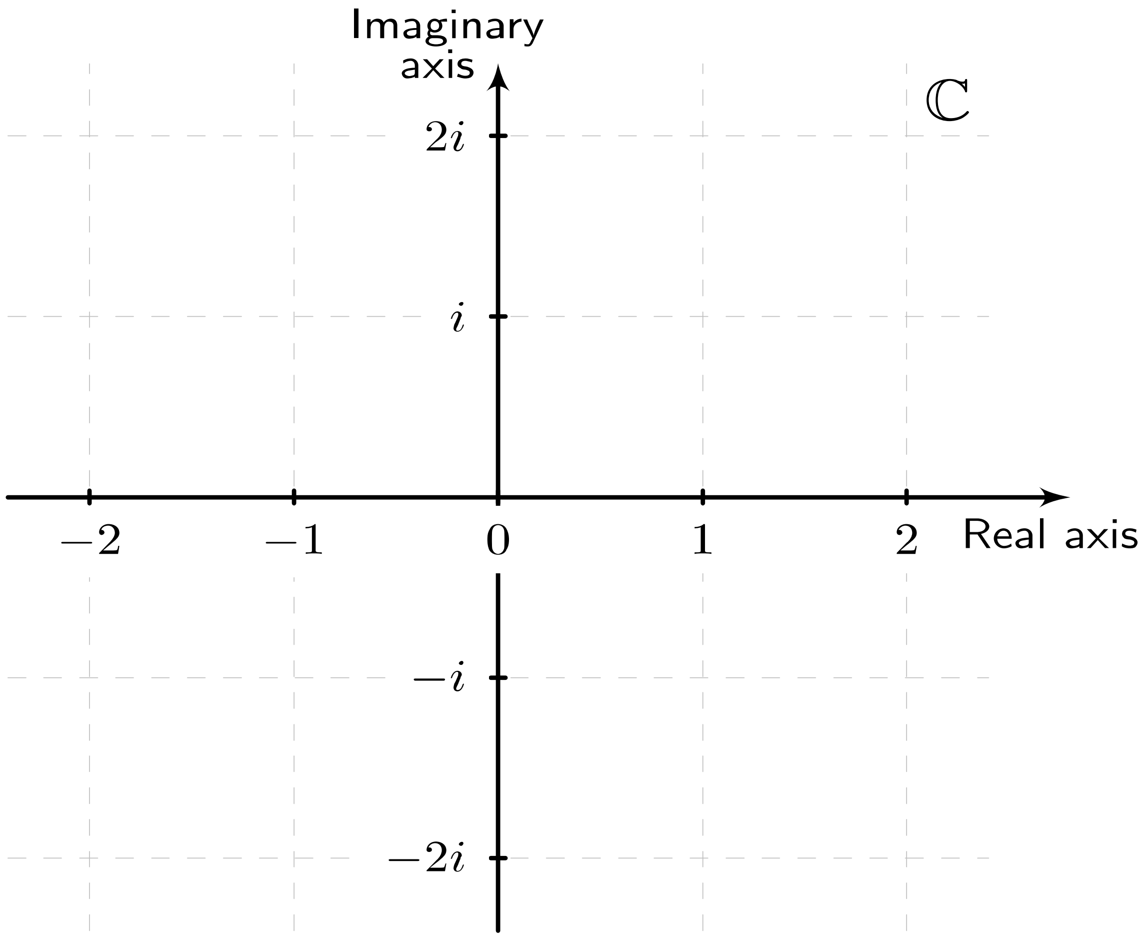
Complex numbers, graphical representation

We can regard complex numbers as points in the plane.



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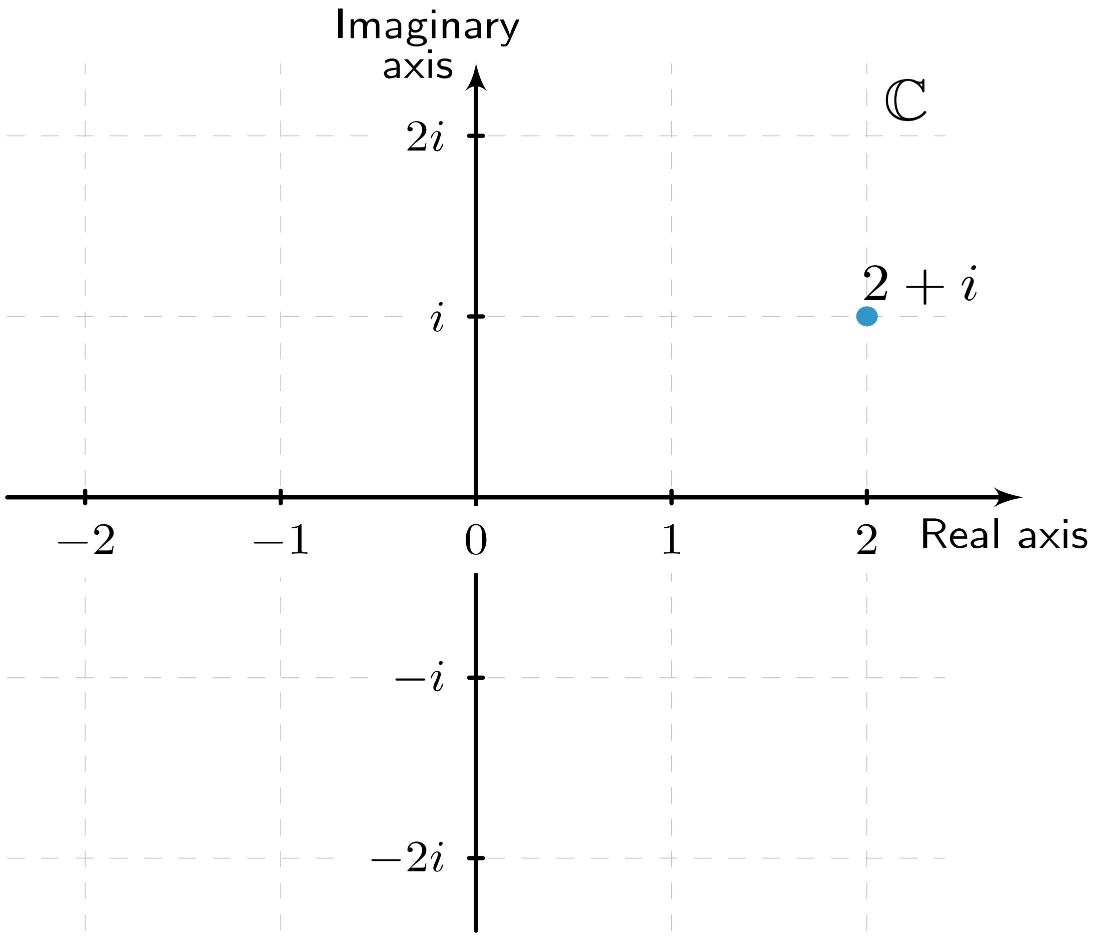

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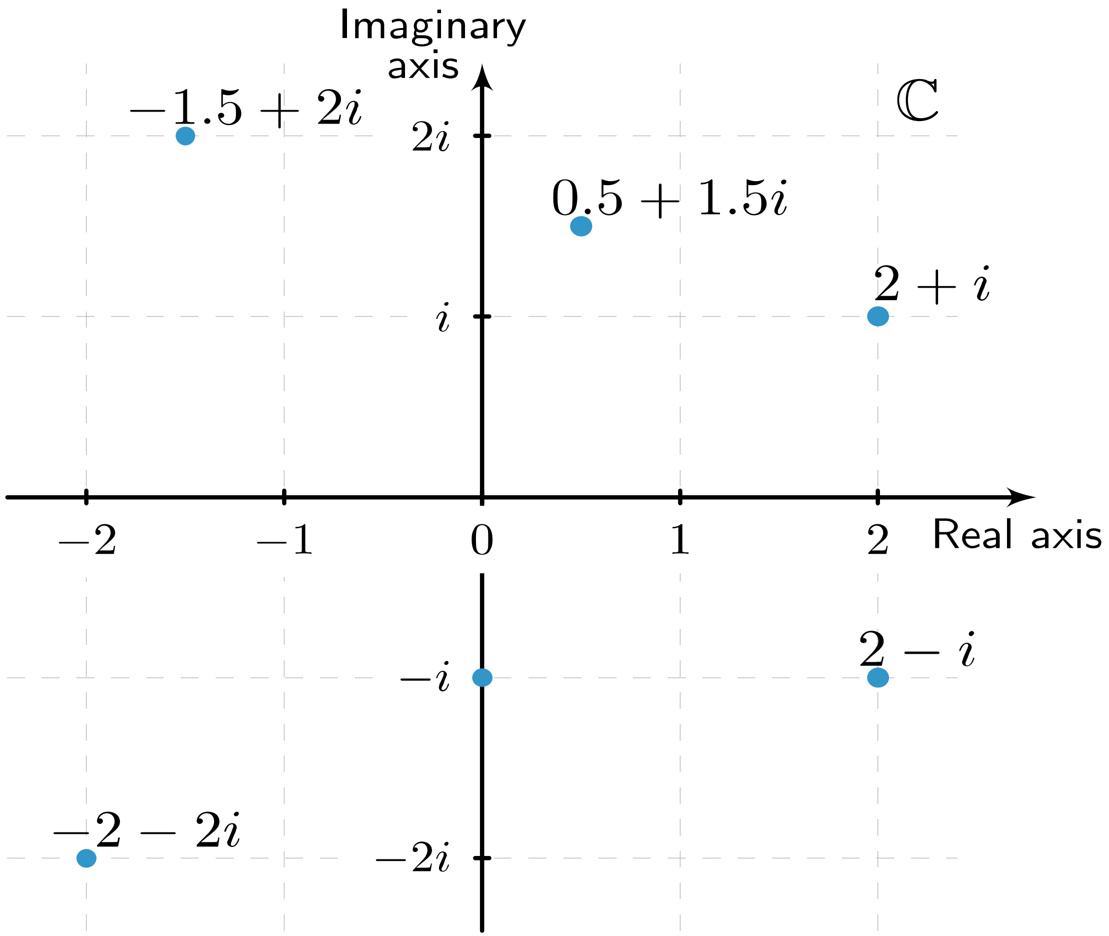

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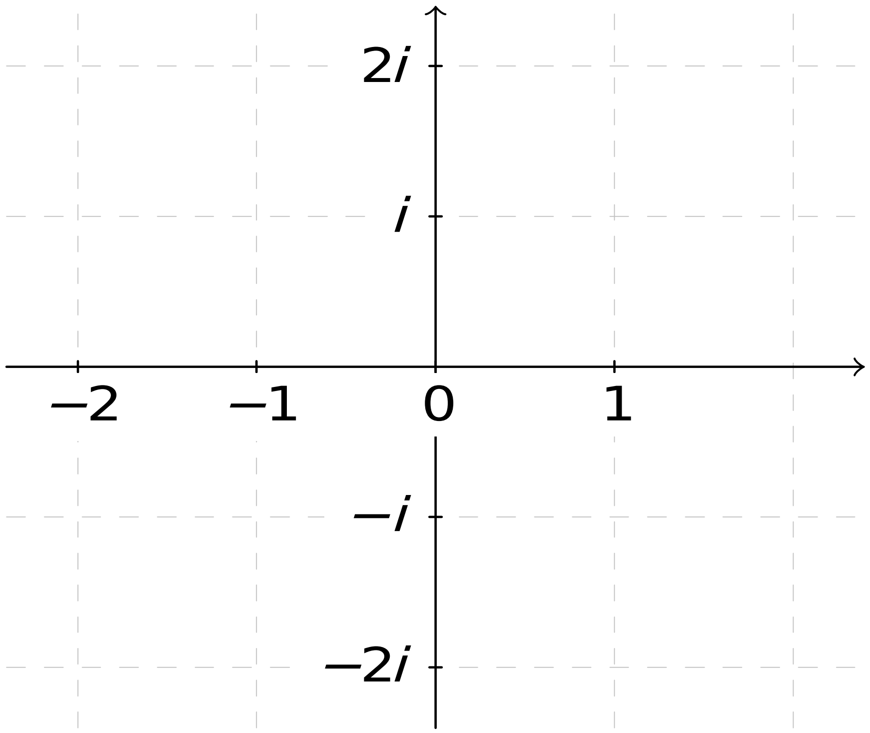

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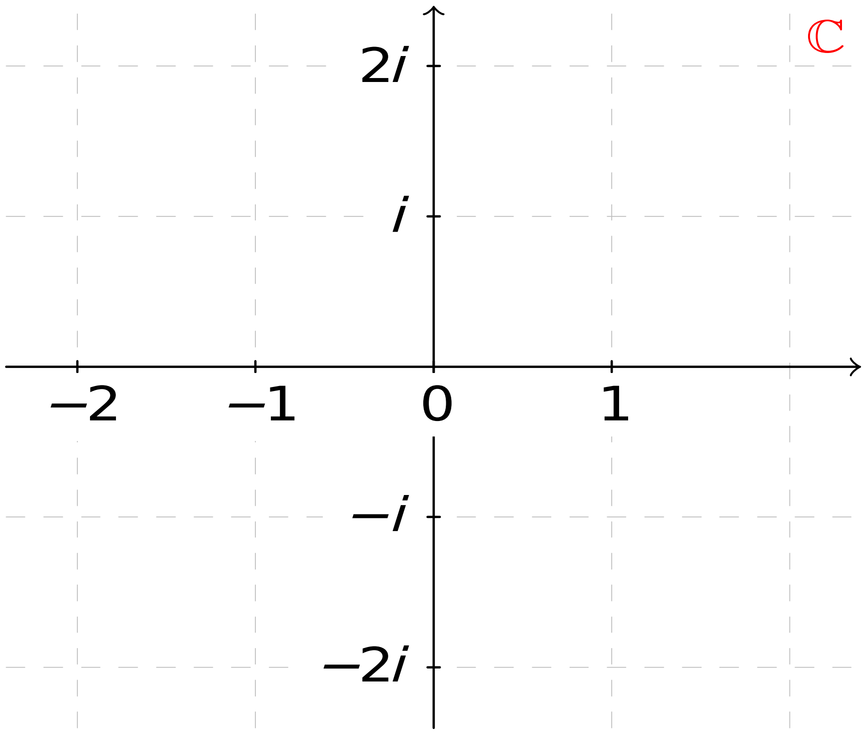

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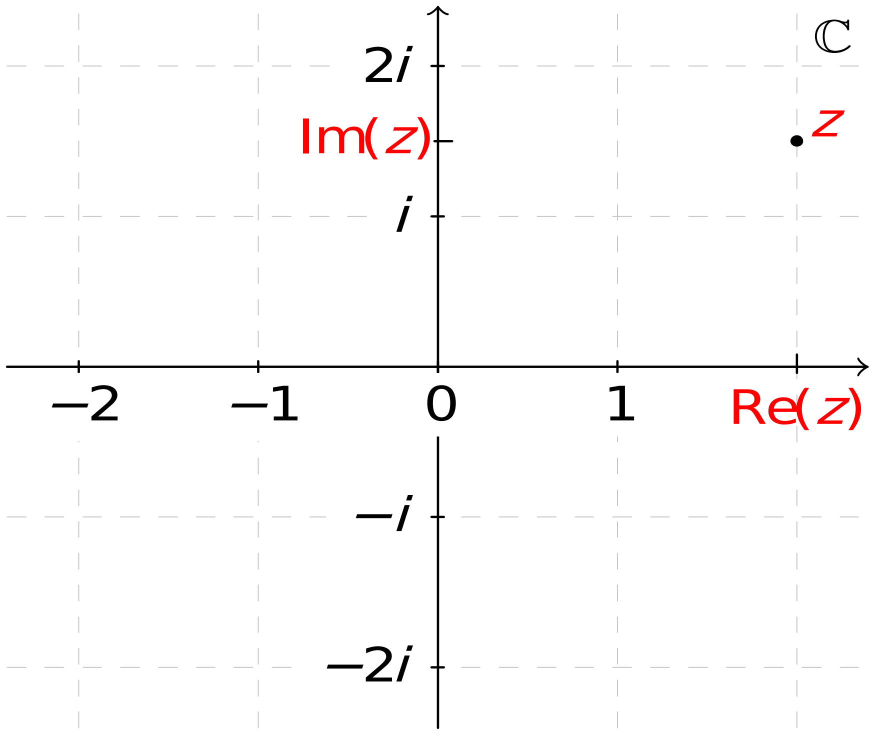

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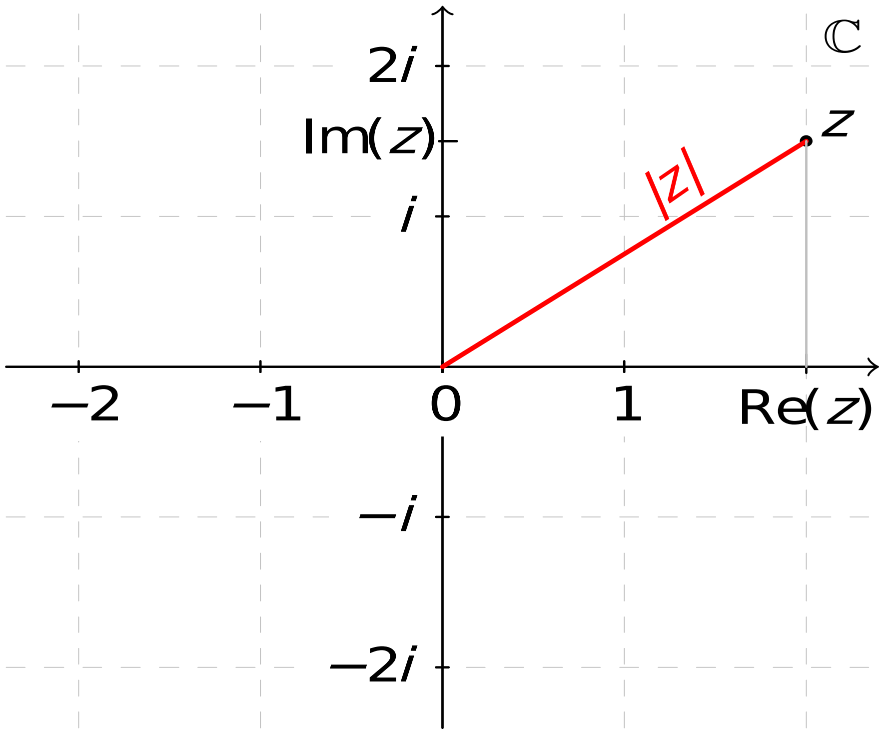

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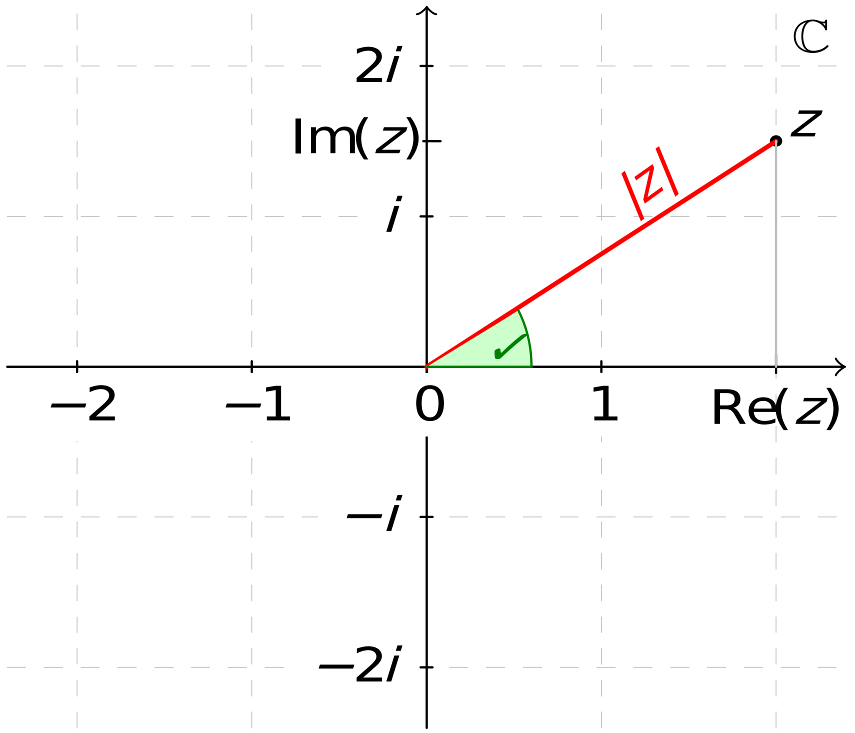

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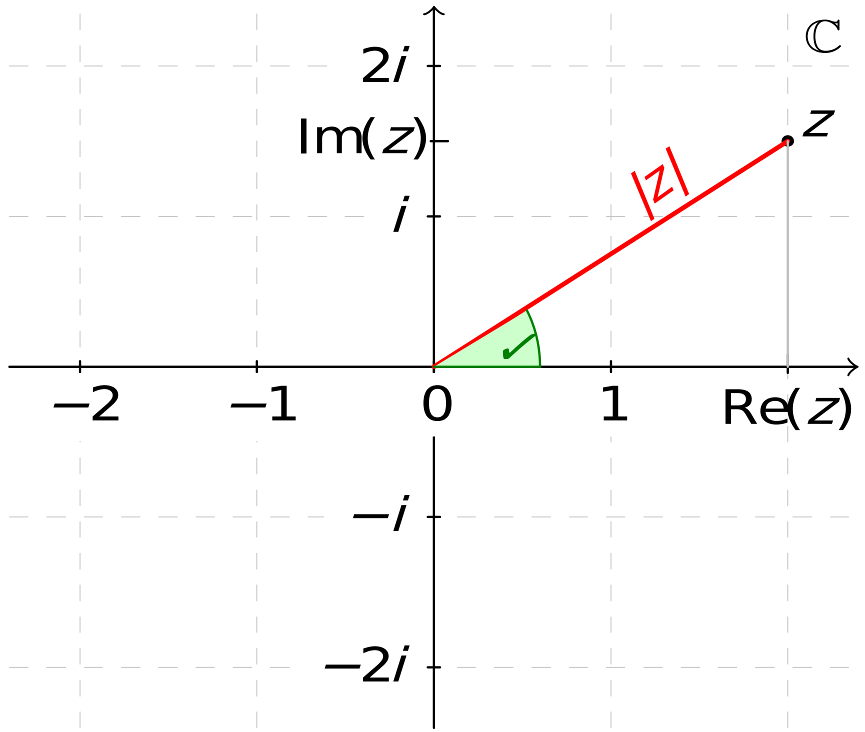

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- Question: What is the value of $|z|$?

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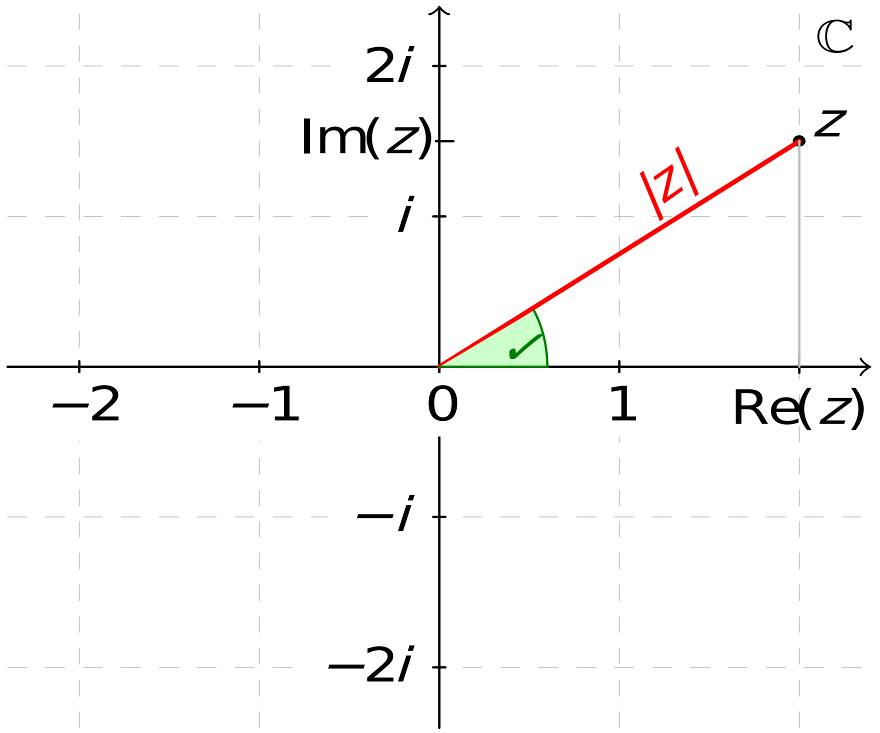

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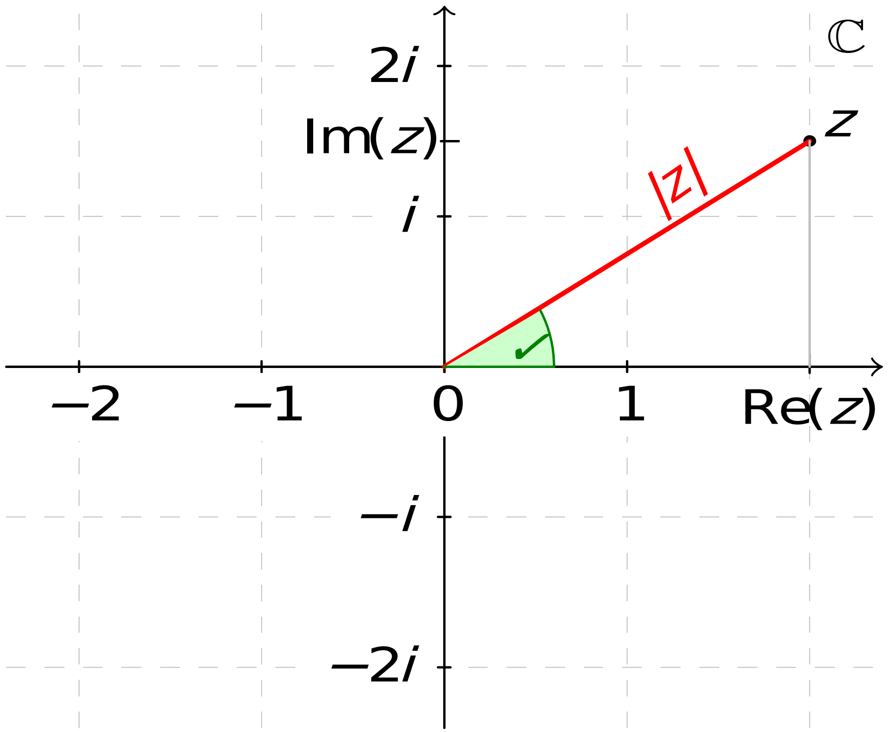

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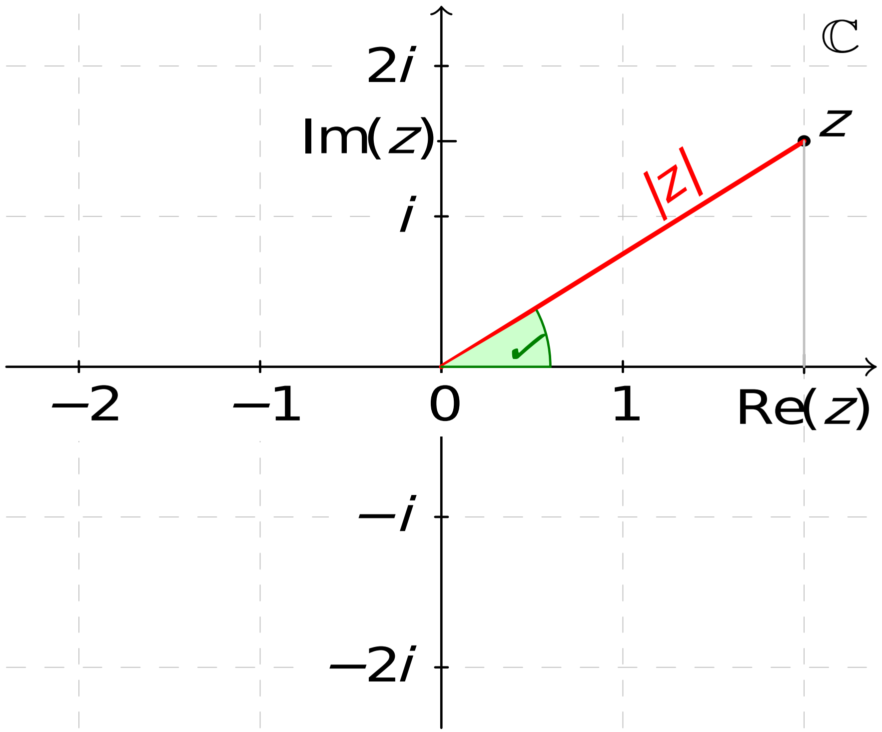

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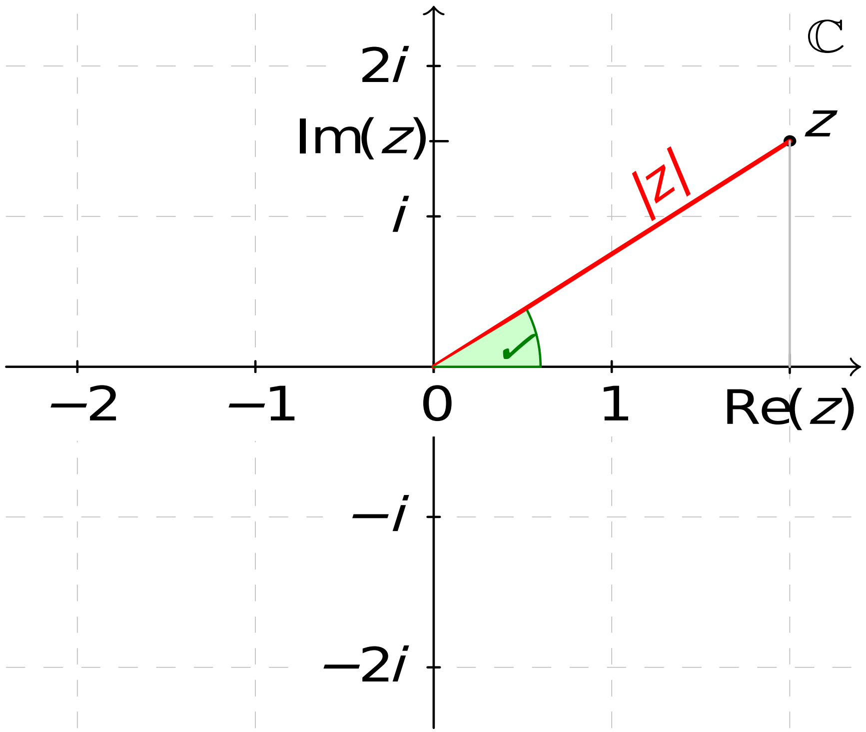

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- Notation: $\arg(z) =$ ✓ argument of z.

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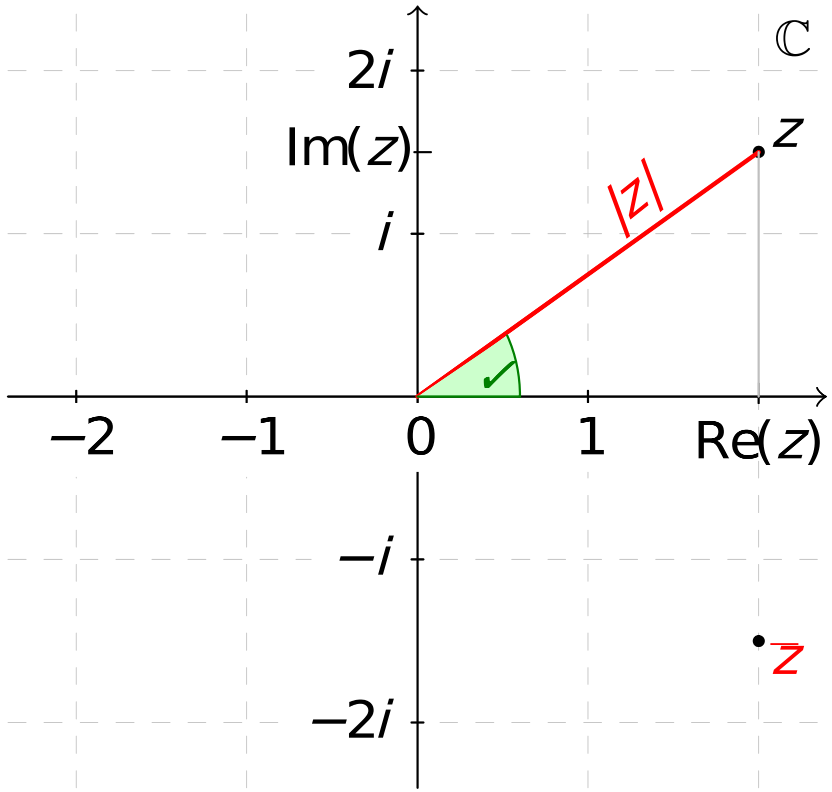

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- $\tan$ ✓ $= \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}$
- Notation: $\arg(z) =$ ✓ argument of z.
- $\bar{z} = \mathrm{Re}(z) - \mathrm{Im}(z)i$.
- $\bar{z}$ is the complex conjugate.

Complex numbers, the absolute value

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- Observe that
$$\begin{aligned} z\,\overline{z} &= (a + b\,i)(a - b\,i) \\ &= a^2 + b \cdot a\,i - a \cdot b\,i - b^2 i^2 \\ &= a^2 + b^2. \end{aligned}$$

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- We have derived the following identity

$$\boxed{|z|^2 = z\,\overline{z}}$$

Complex numbers, division

With the complex conjugate we can perform division.

- $$\frac{3+4i}{7-2i} = \left[\frac{3+4i}{7-2i}\right] \cdot \left[\frac{7+2i}{7+2i}\right]$$

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  $$= \frac{13+34i}{53}$$

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- Therefore
$$\frac{z_1}{z_2} = \frac{z_1\cdot\overline{z_2}}{|z_2|^2}$$

Complex numbers, division

If $z = i$ then $\frac{1}{z}$ is equal to

(a) i

(b) $-i$

(c) -1

(d) $1 - i$

Complex numbers, the quadratic formula

Theorem

If $a \neq 0$, b *and* c *are arbitrary real numbers, then the solutions of* $az^2 + bz + c = 0$ *are given by*

$$\boxed{z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

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- The formula is called the **quadratic formula**.
- If $b^2 - 4ac \geq 0$ then the quadratic formula yields one or two *real* numbers.
- If $b^2 - 4ac < 0$ then the quadratic formula yields two non-real *complex* numbers:

$$z_{1,2} = \frac{-b \pm i\sqrt{|b^2 - 4ac|}}{2a}.$$

Complex numbers, the quadratic formula

- The solutions of $z^2 + 2z + 10 = 0$ are

$$z_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4 * 1 * 10}}{2 * 1}$$

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$$z_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4 * 1 * 10}}{2 * 1}$$
$$= \frac{-2 \pm \sqrt{-36}}{2}$$

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  in other words: $z_1 = -1 + 3i$ and $z_2 = -1 - 3i$.

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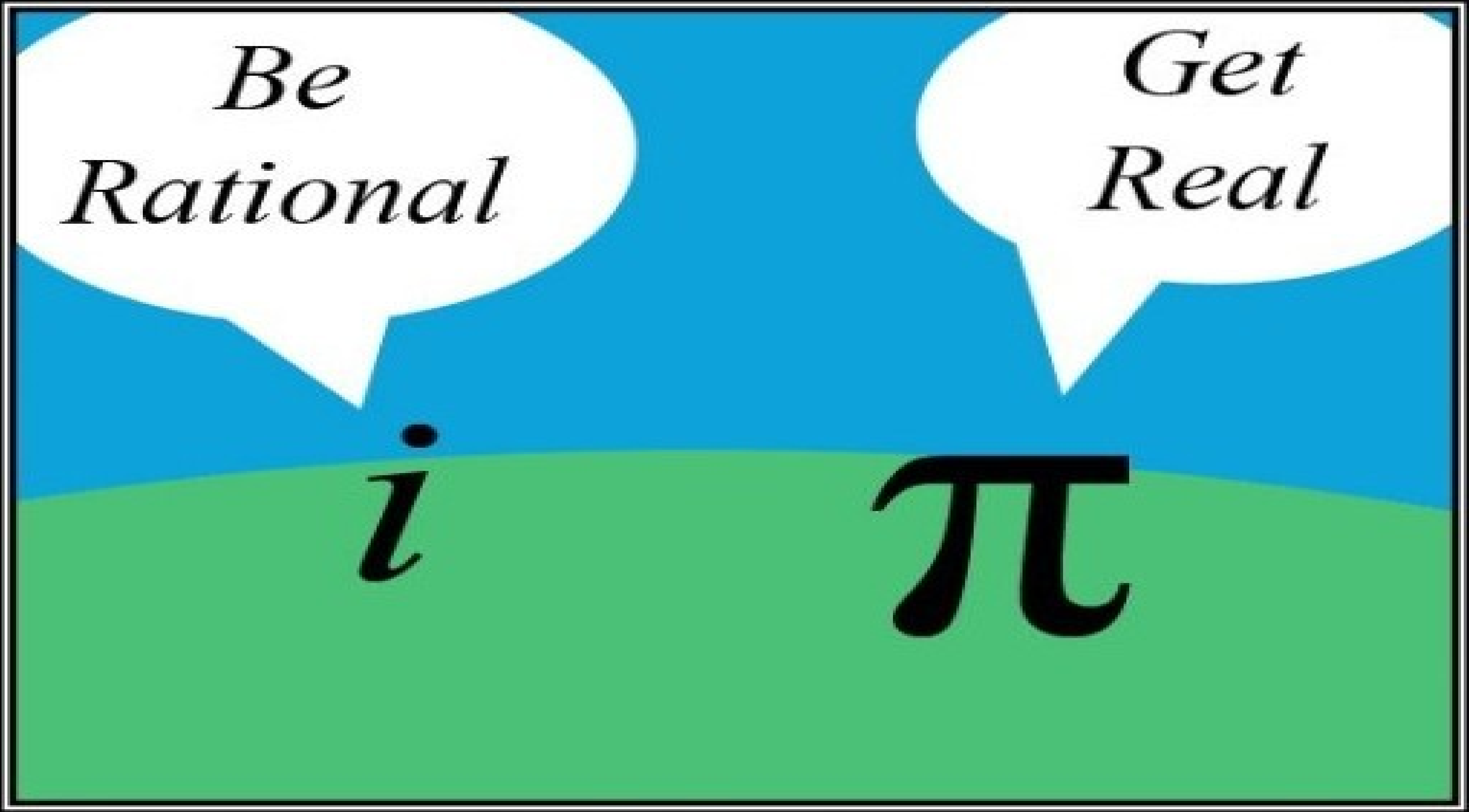
- Note that $\overline{z_1} = z_2$.


Be Rational
Get Real
i
π

Complex numbers, Eulers formula

Definition

The identity

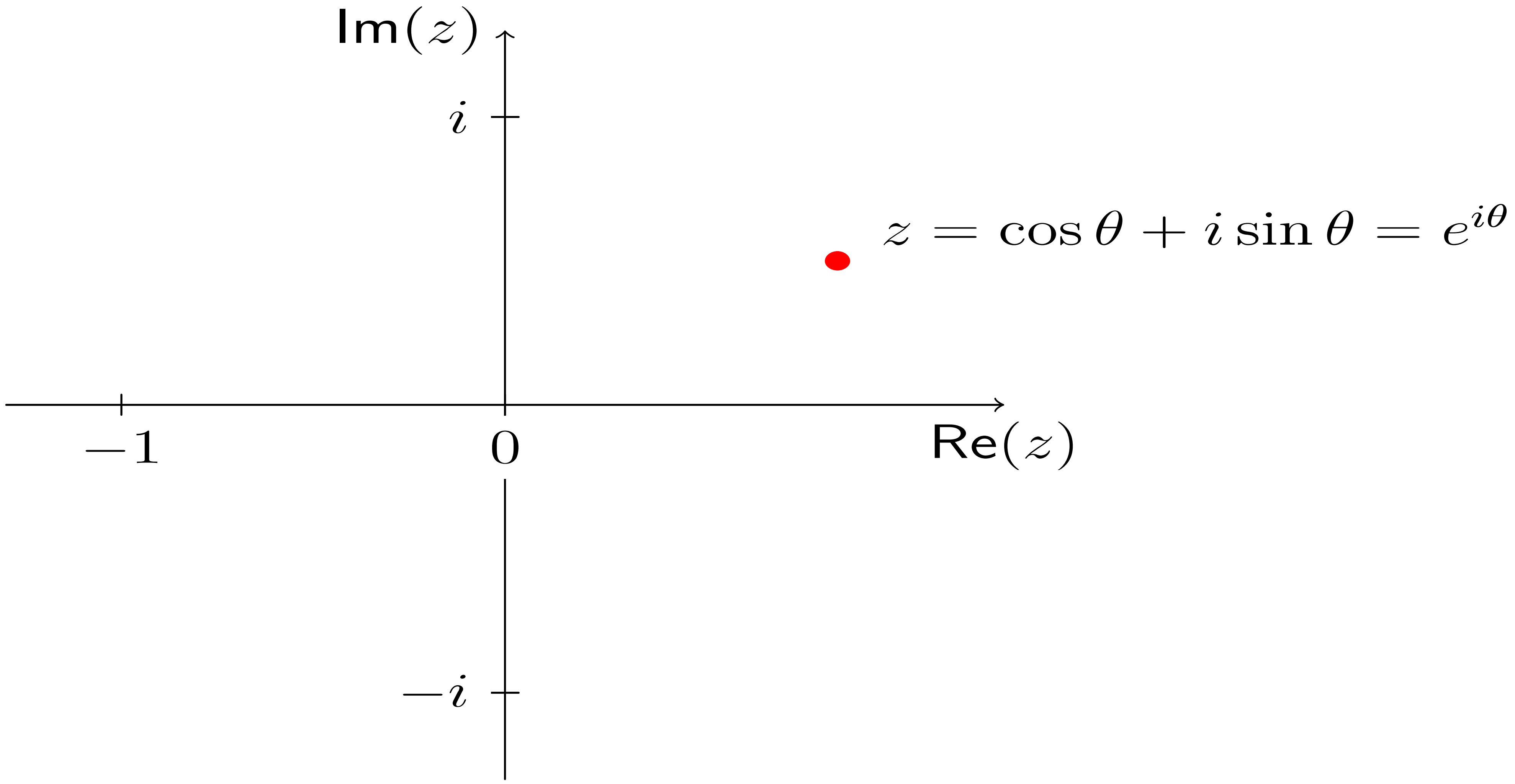
$$\boxed{e^{i\theta} = \cos\theta + i\,\sin\theta}$$

is called Eulers formula.

Lemma

- $$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i};$$
- $$e^{i\theta} e^{i\varphi} = e^{i(\theta+\varphi)}.$$
- $$\frac{1}{e^{i\theta}} = \overline{e^{i\theta}} = e^{-i\theta}.$$

Location $e^{i\theta}$



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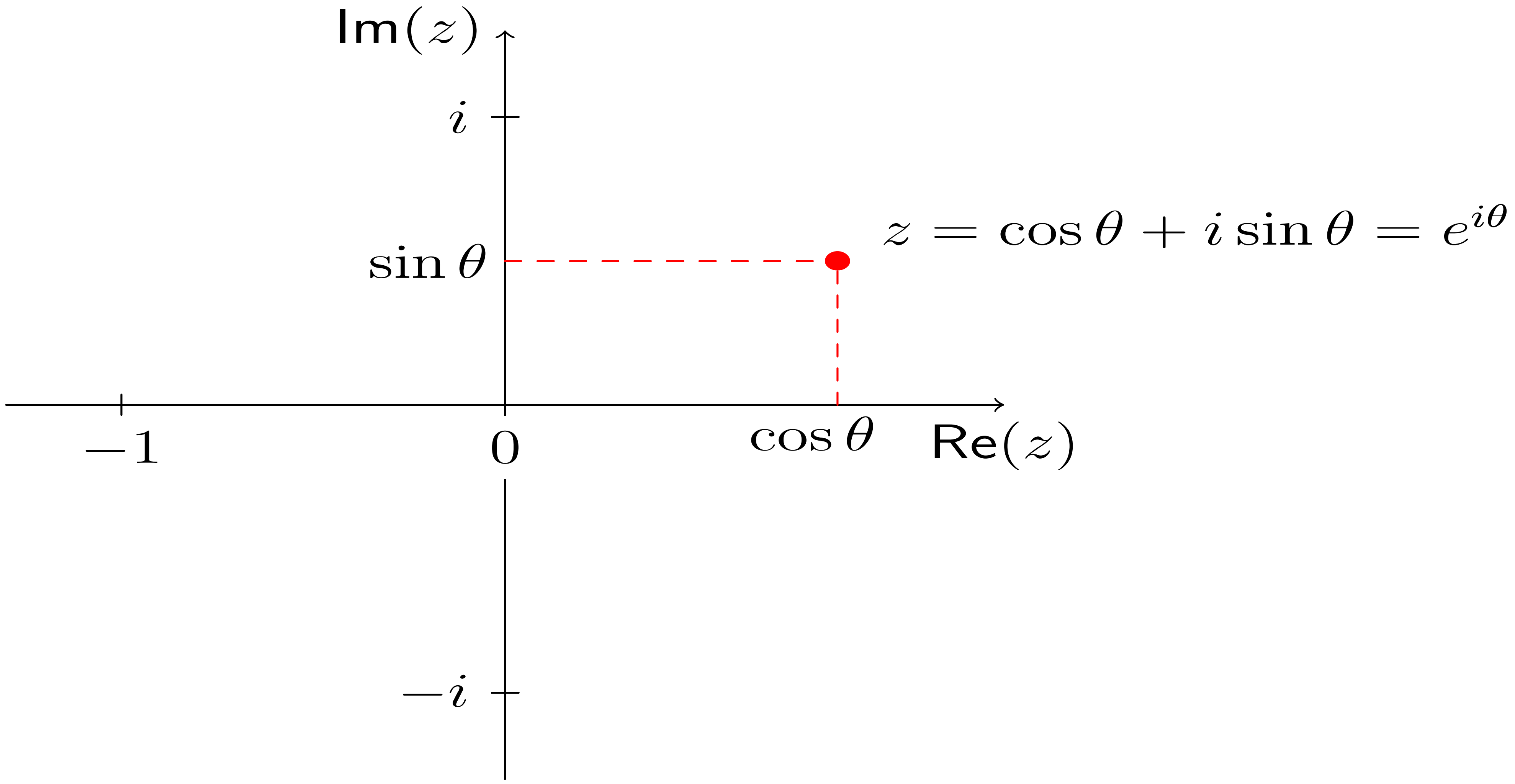

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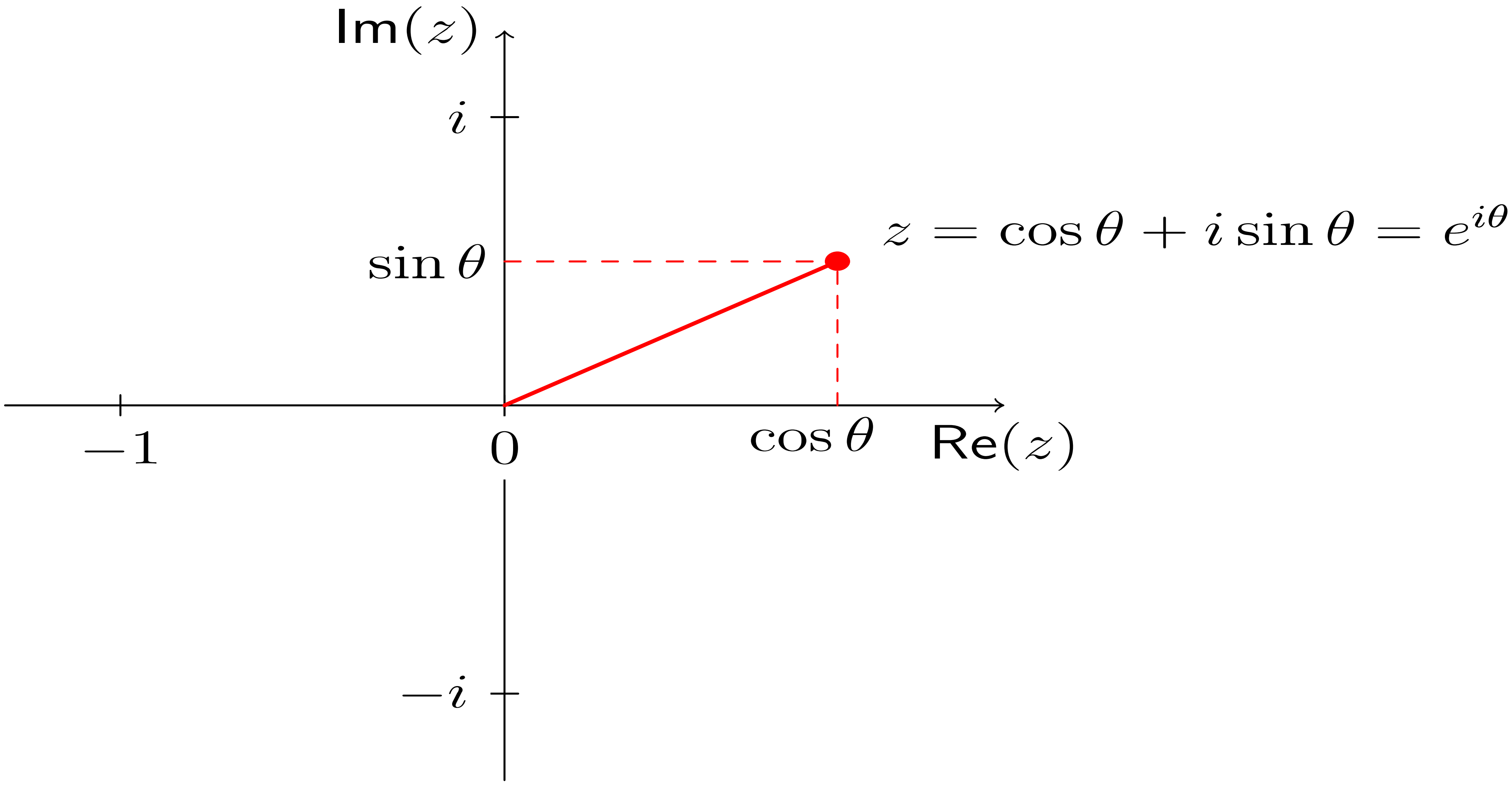

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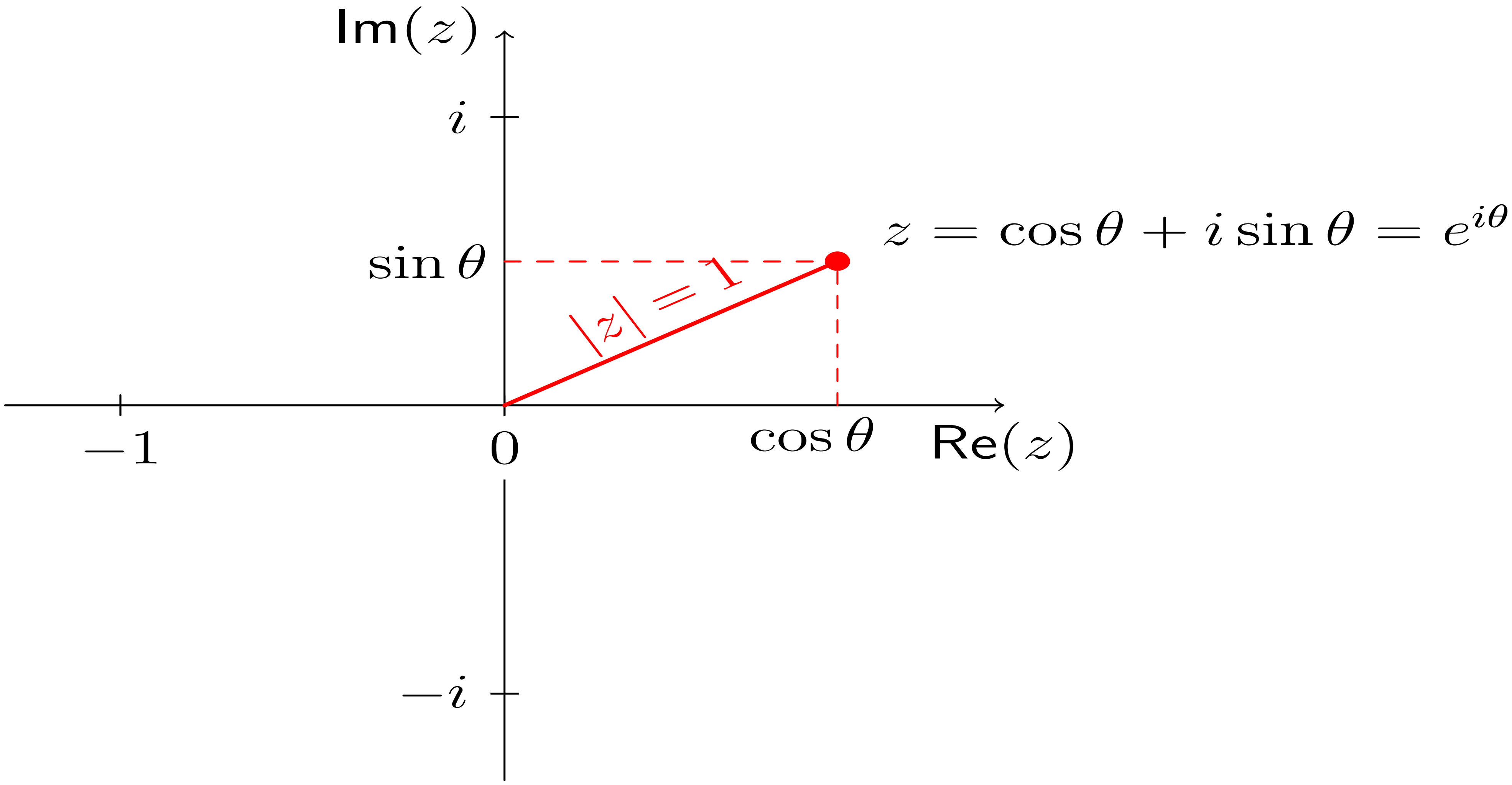

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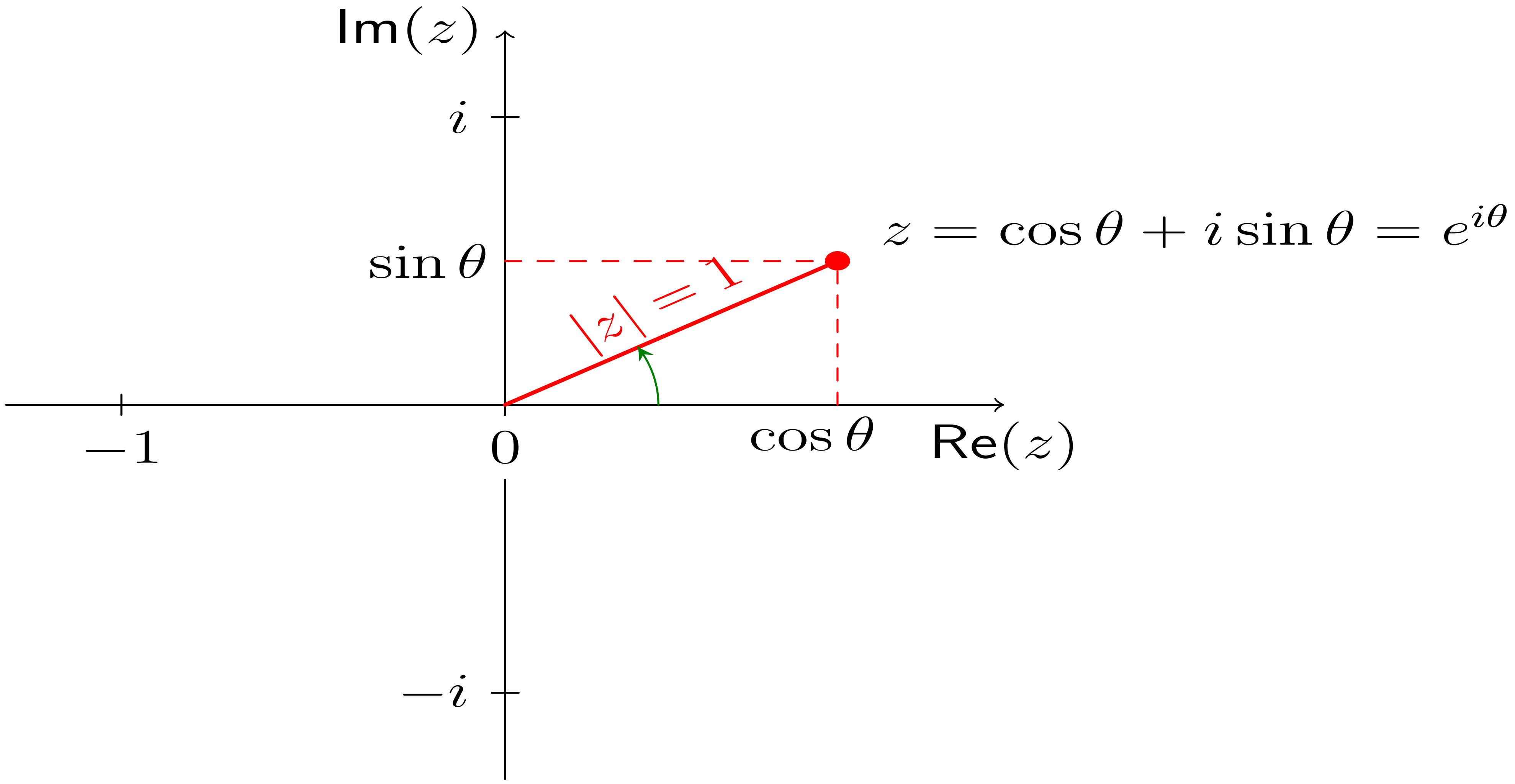

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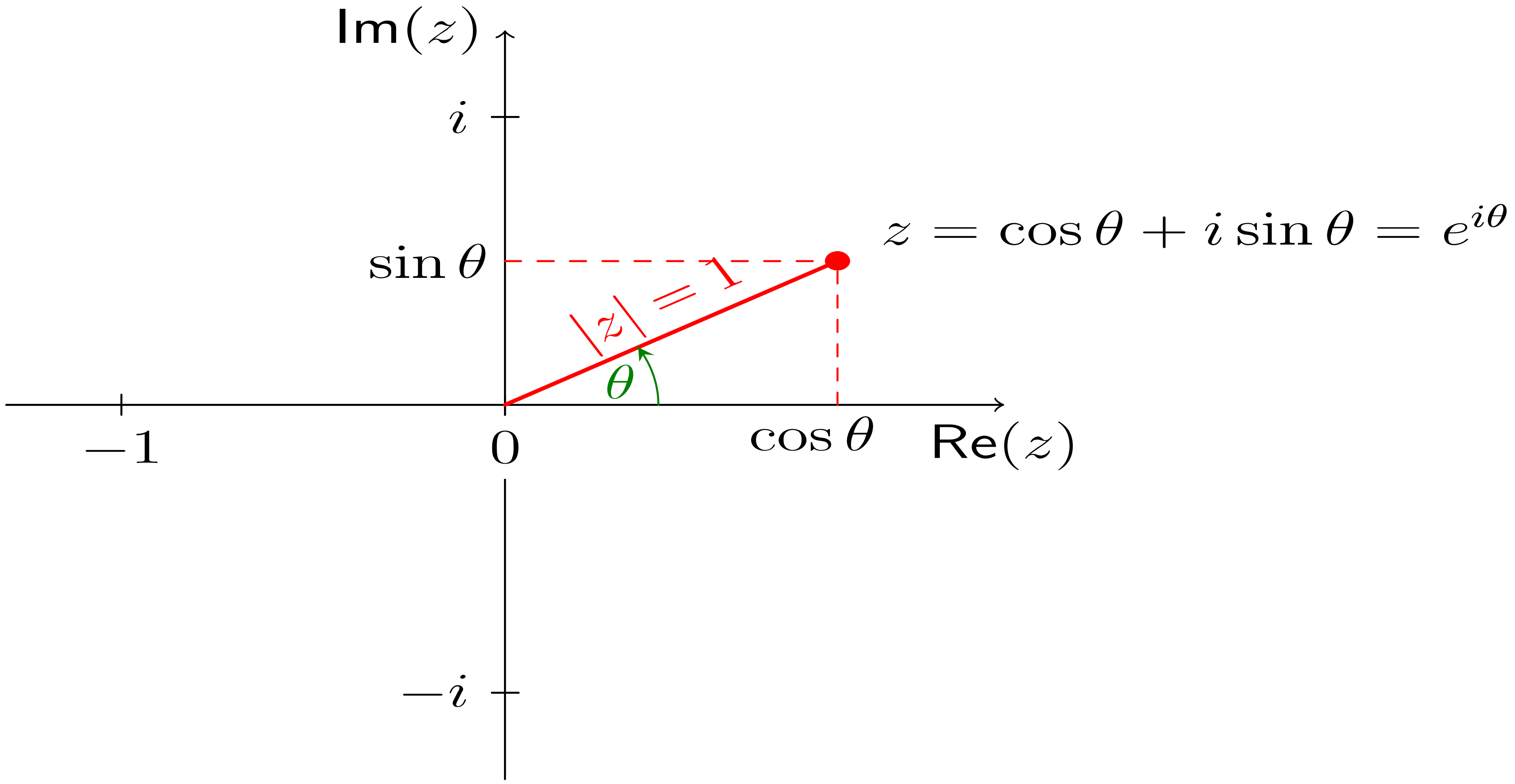

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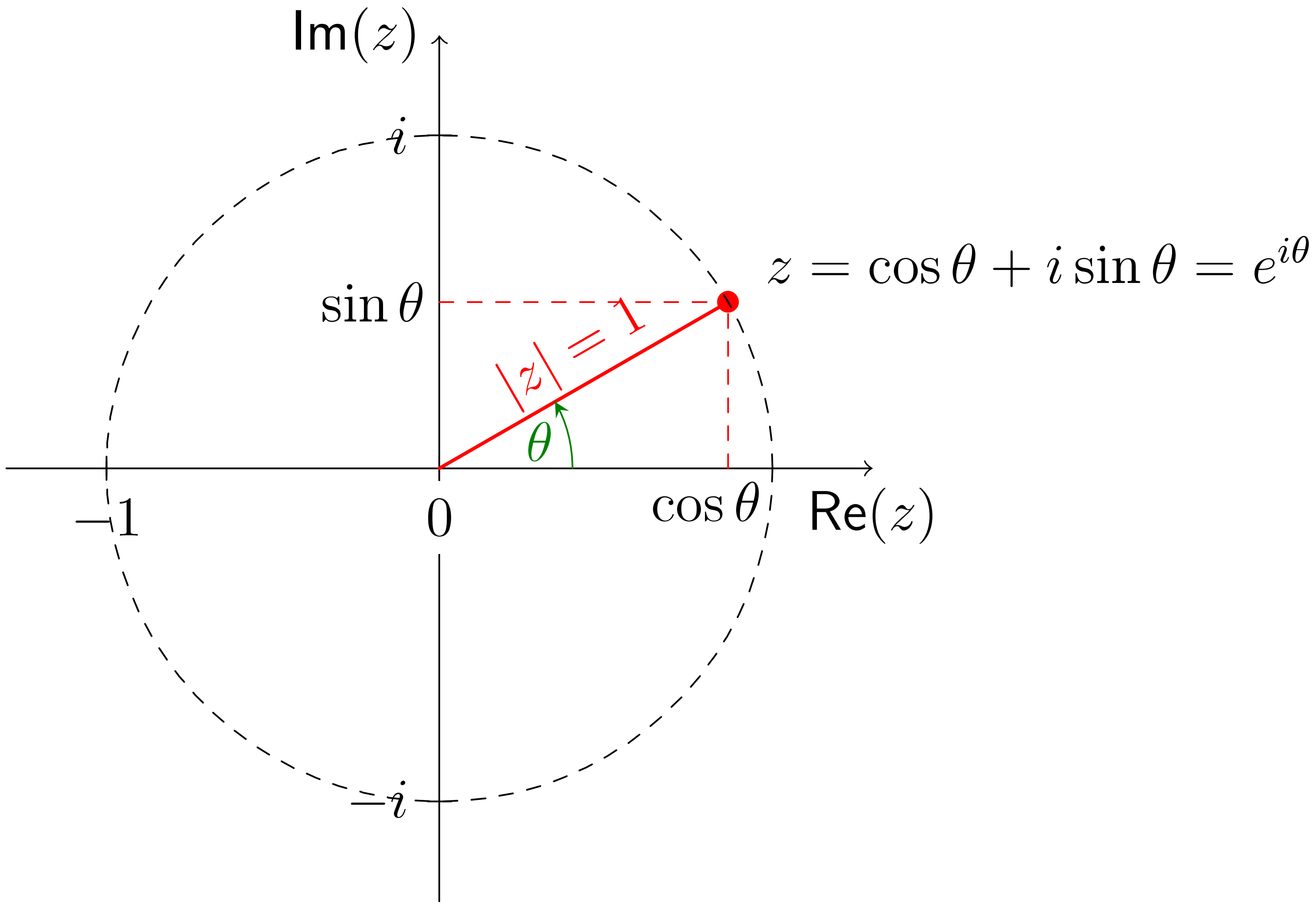

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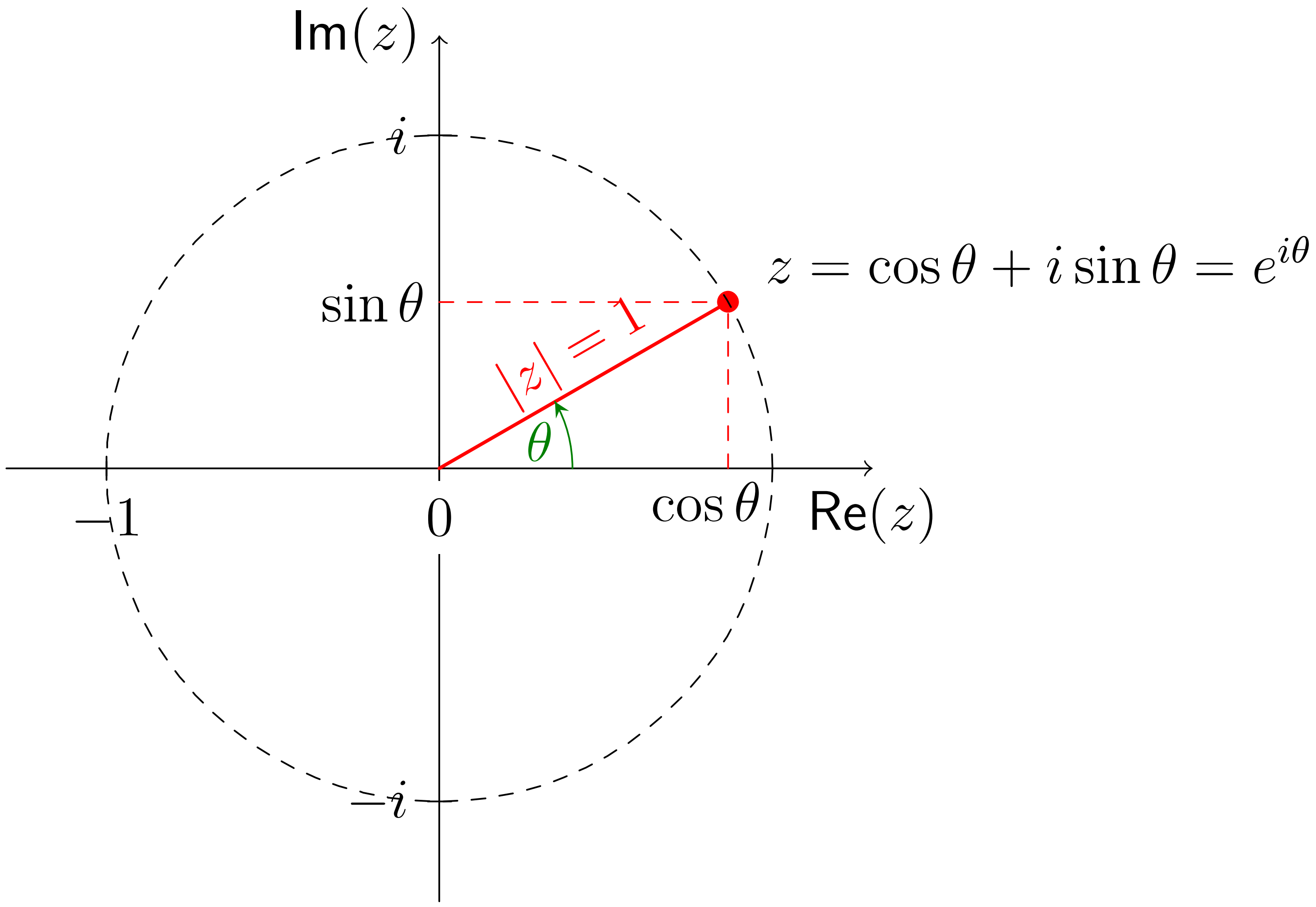

Eulers Identity: $e^{i\pi} + 1 = 0.$

Complex numbers, complex exponential

Question: where is the complex number $z = re^{i\theta}$ located in the complex plane $\mathbb{C}$?

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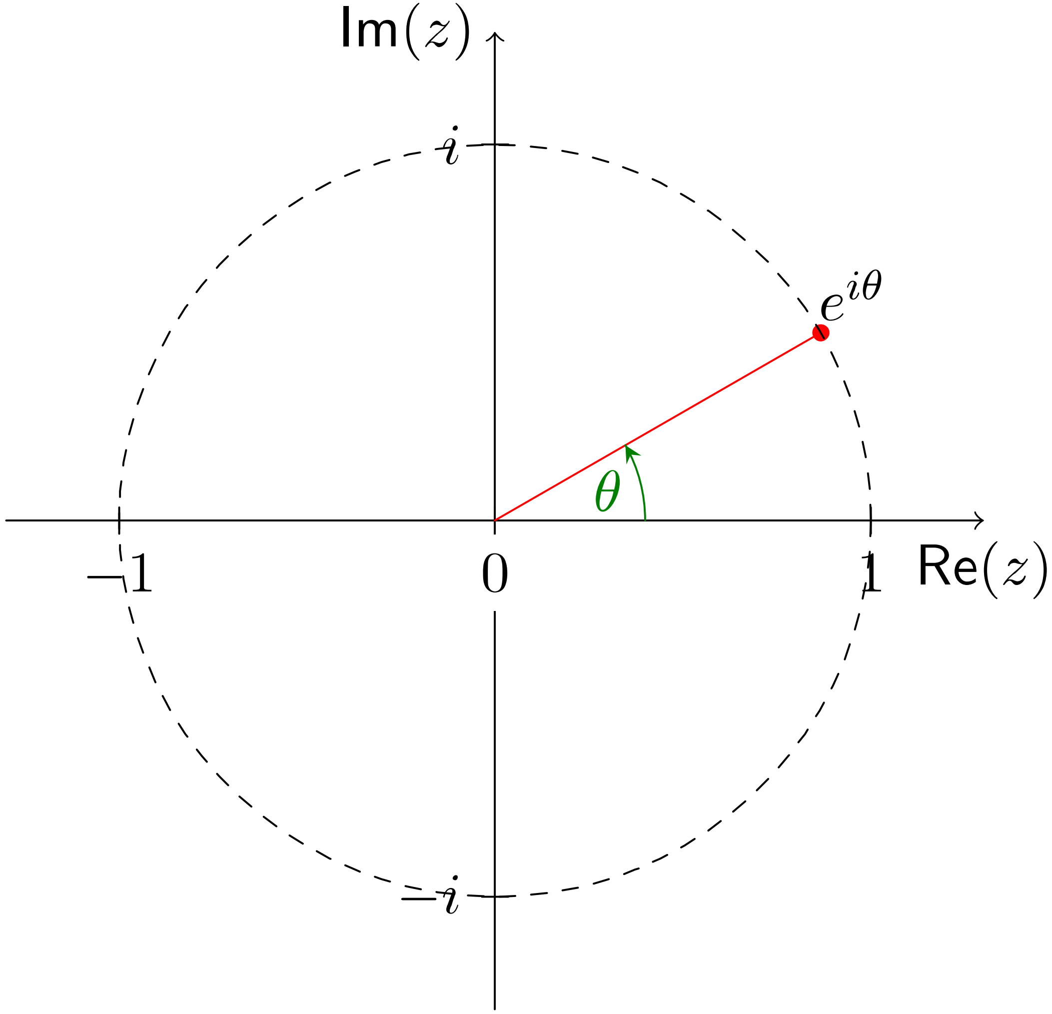

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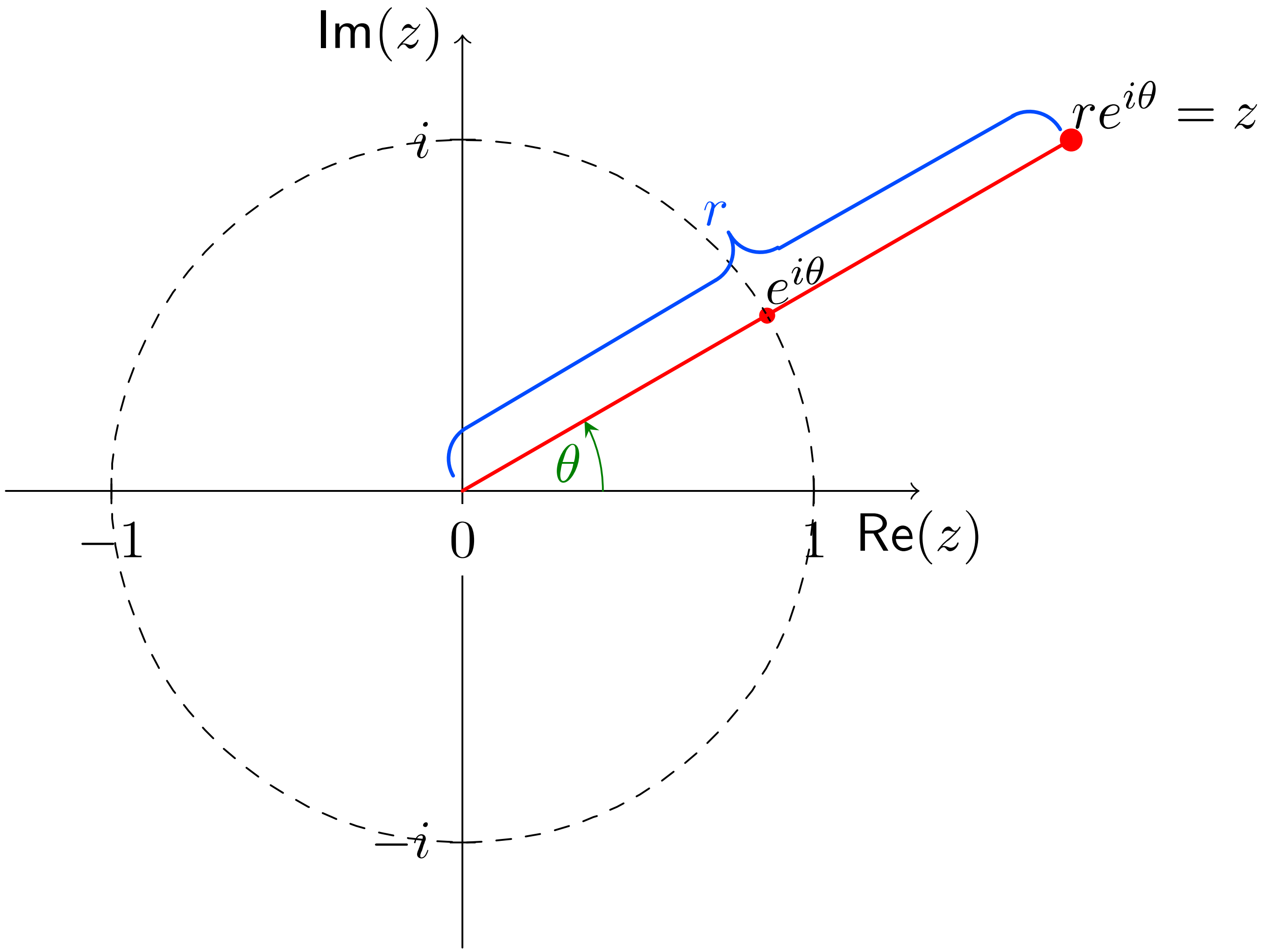

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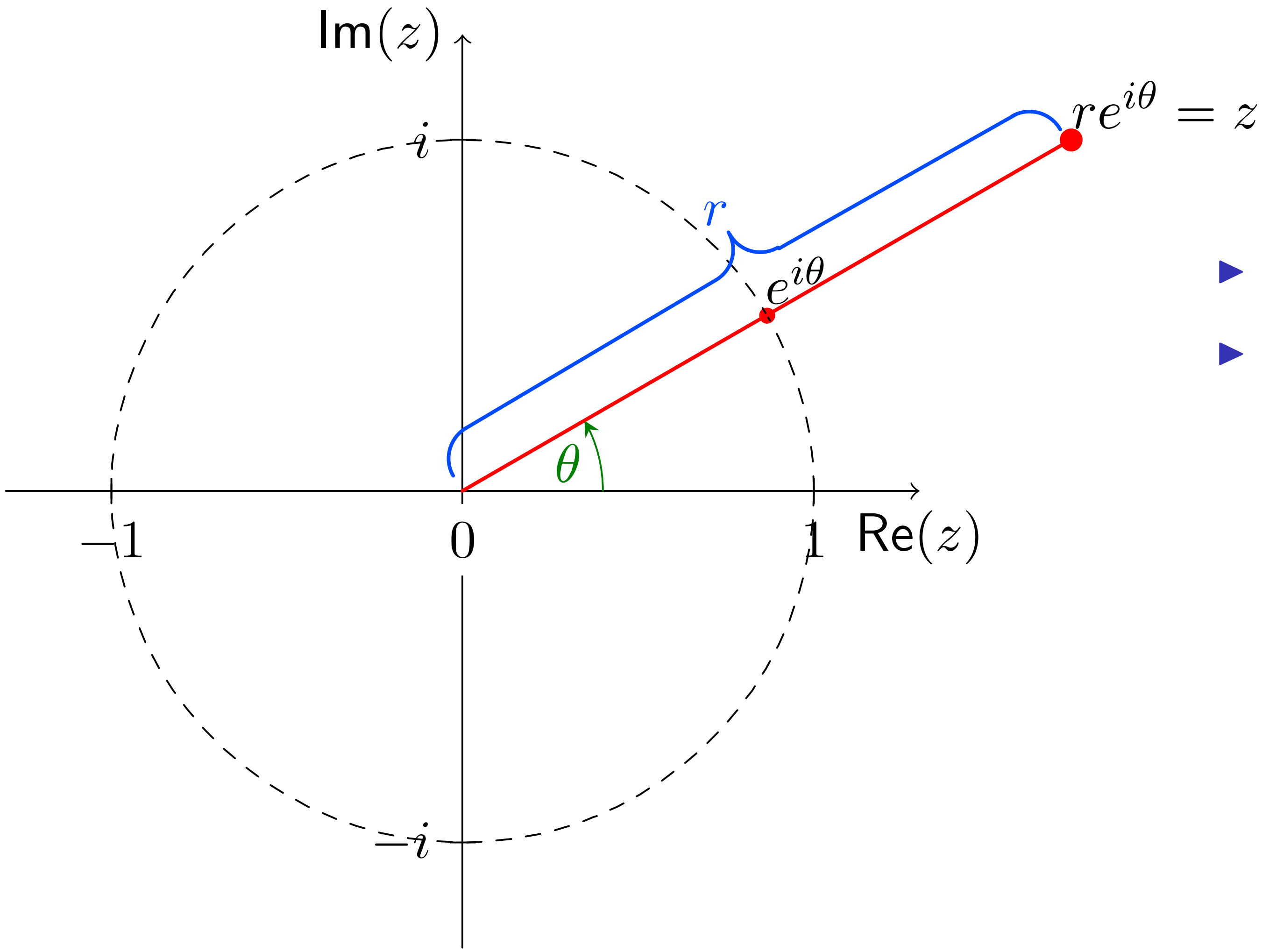

- $r = |z|$
- $\arg(z) = \theta$

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$$z = |z|e^{i\,\arg(z)}.$$

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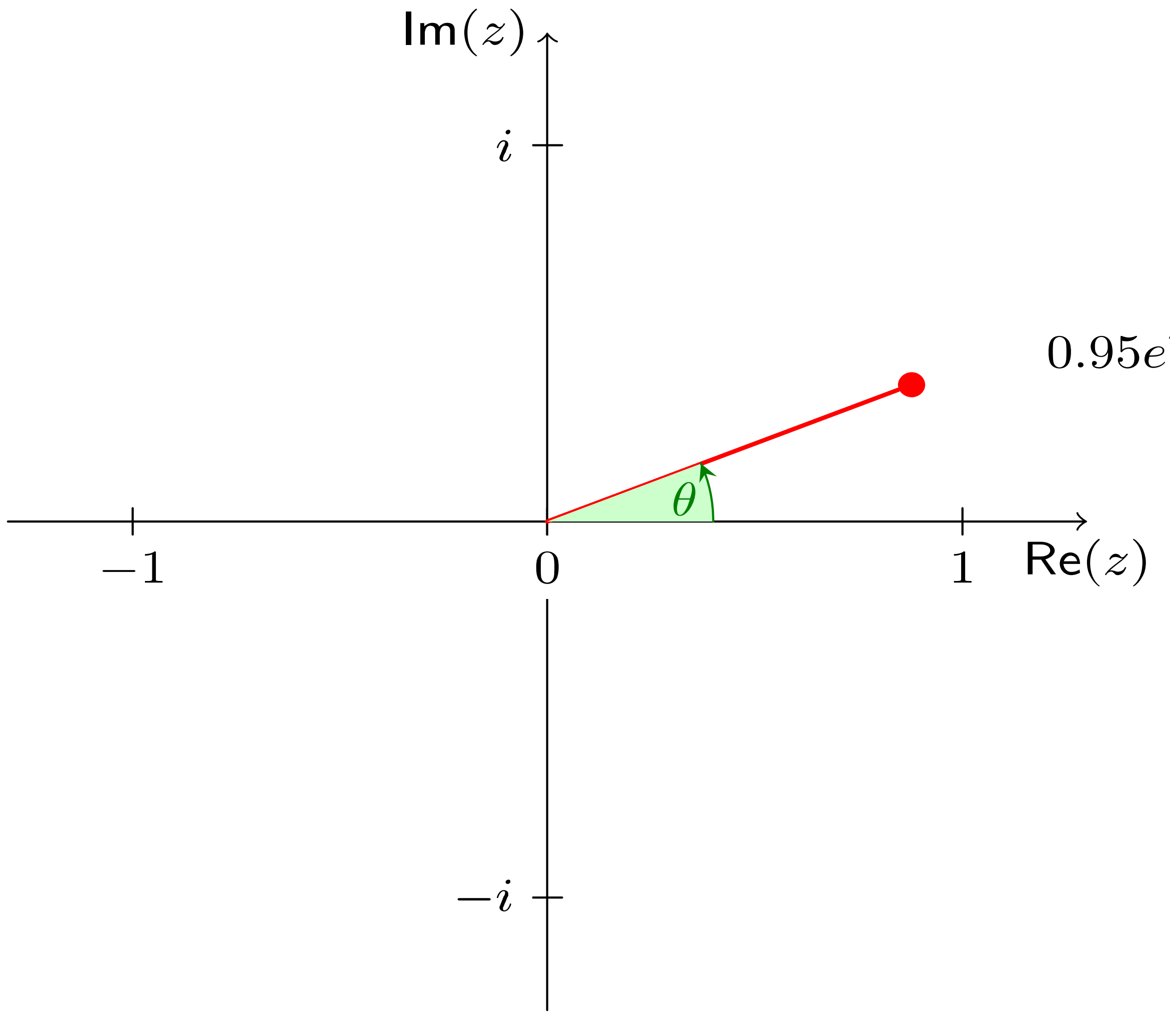
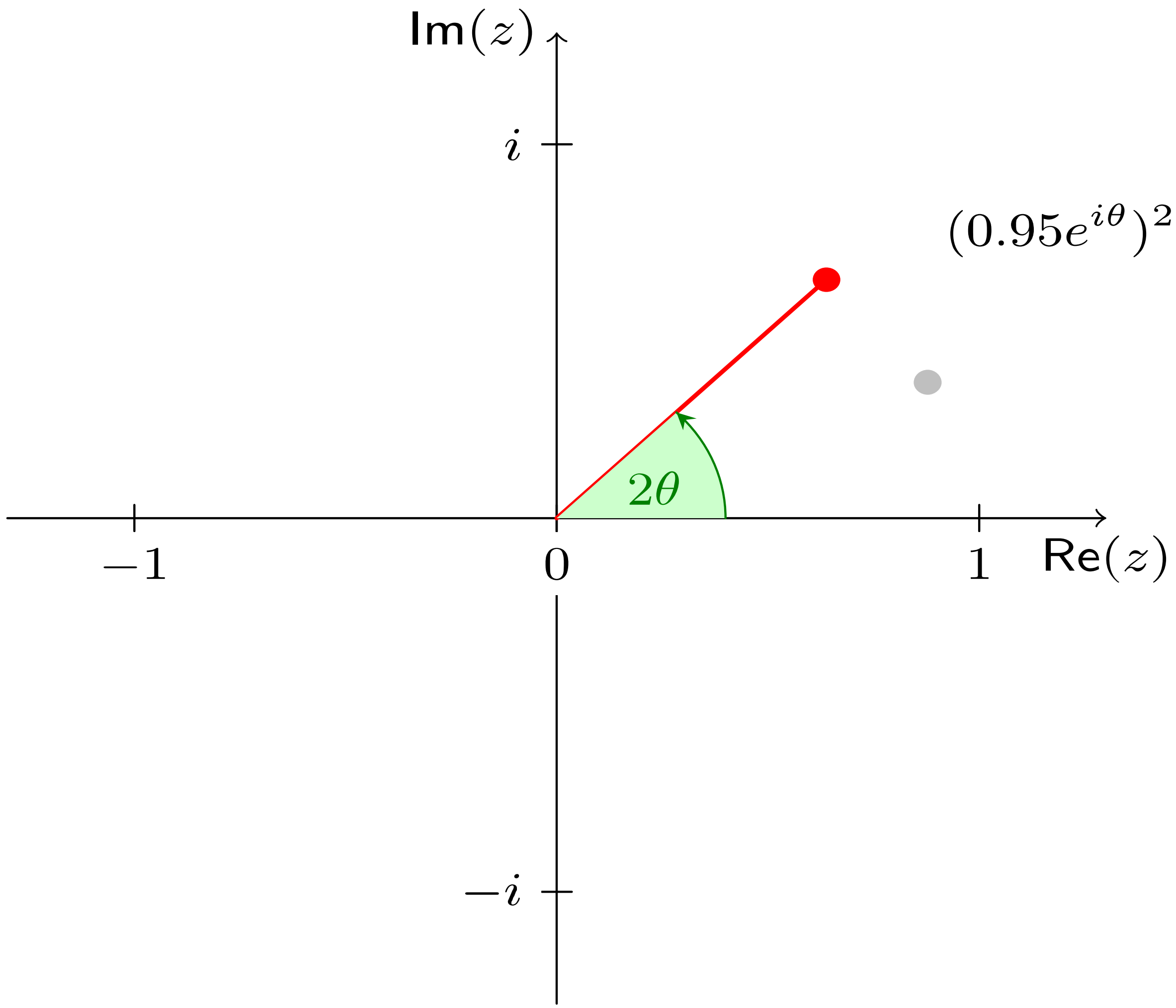
Complex numbers, powers



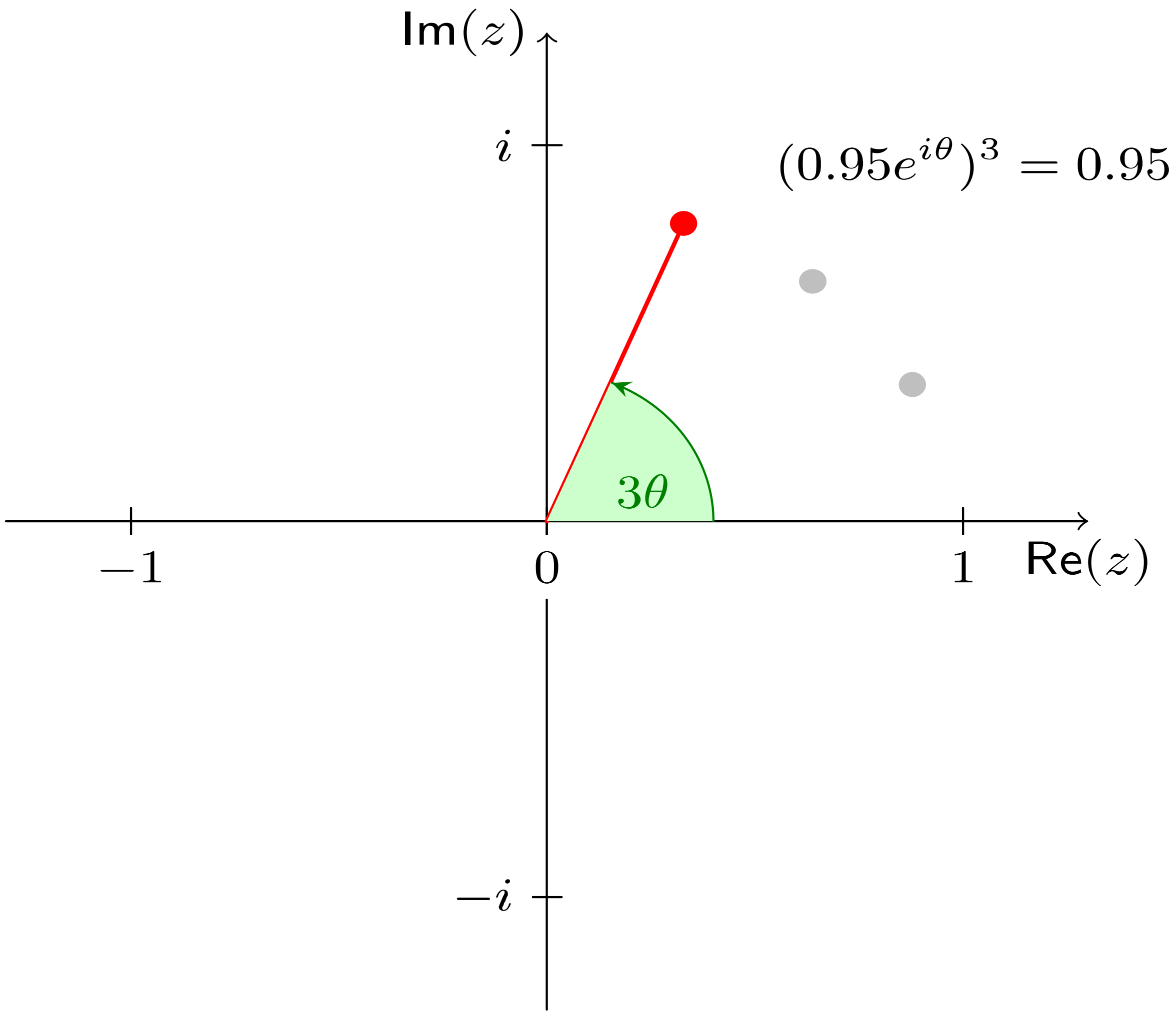
$$0.95e^{i\theta} = 0.95[\cos\theta + i\sin\theta]$$

Complex numbers, powers



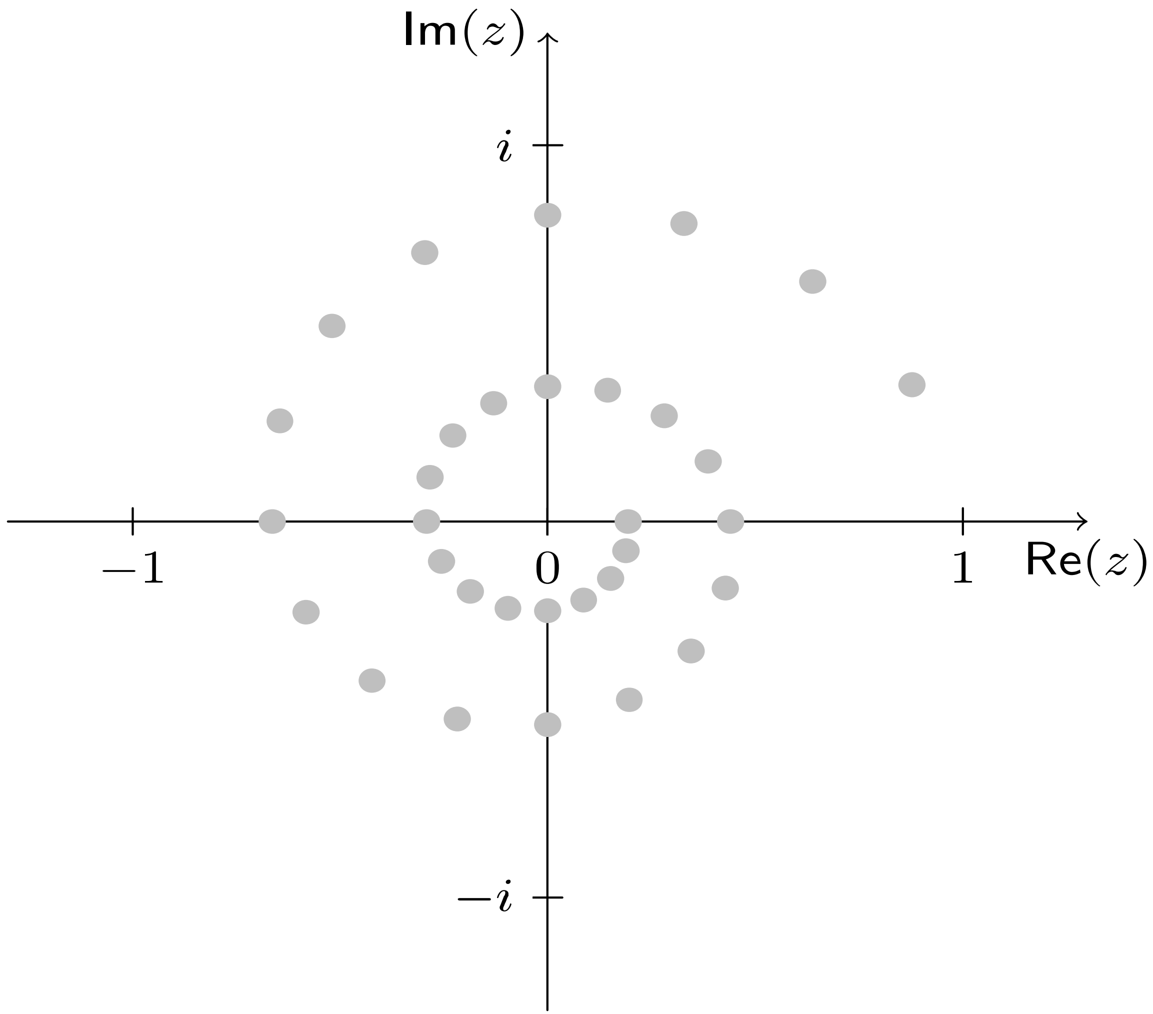
$$(0.95e^{i\theta})^2 = 0.95^2[\cos(2\theta) + i\sin(2\theta)]$$

Complex numbers, powers



$(0.95e^{i\theta})^3 = 0.95^3[\cos(3\theta) + i\sin(3\theta)]$

Complex numbers, powers



De Moivres Theorem

- We see that

$$(\cos\theta + i\sin\theta)^2 = e^{i\theta}e^{i\theta} = e^{i2\theta} = \cos(2\theta) + i\sin(2\theta)$$

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- This leads to **De Moivre's Theorem:**

Theorem

For $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}$ we have

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Quotients

Example

Divide

$$z_1 = 4e^{i\pi/3} \quad \text{by} \quad z_2 = 5e^{i\pi/4}$$

- $$\frac{z_1}{z_2} = \frac{4e^{i\pi/3}}{5e^{i\pi/4}} = \frac{4e^{i\pi/3}}{5}e^{-i\pi/4} = \frac{4}{5}e^{i(\pi/3-\pi/4)} = \frac{4}{5}e^{i\pi/12}.$$

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- Remember:
  - absolute values are divided,
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Finding cube roots

Find all complex numbers z such that $z^3 = 8$.

- Write $z = re^{i\theta}$, then

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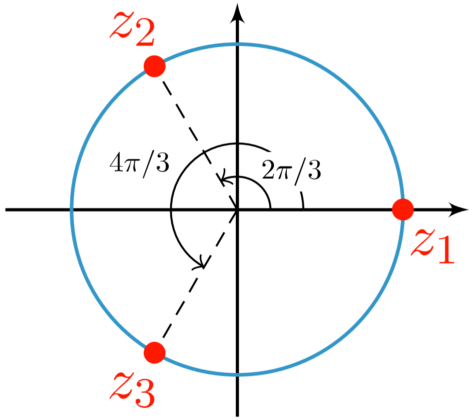
- The solutions of $z^3 = 8$ are
$$z_1 = 2e^{i0} = 2,$$
$$z_2 = 2e^{i2\pi/3} = -1 + i\sqrt{3},$$
$$z_3 = 2e^{i4\pi/3} = -1 - i\sqrt{3}.$$



Finding n-th roots

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Let n be a positive integer and let w be an arbitrary complex number, unequal to 0. Find all z such that $z^n = u$.

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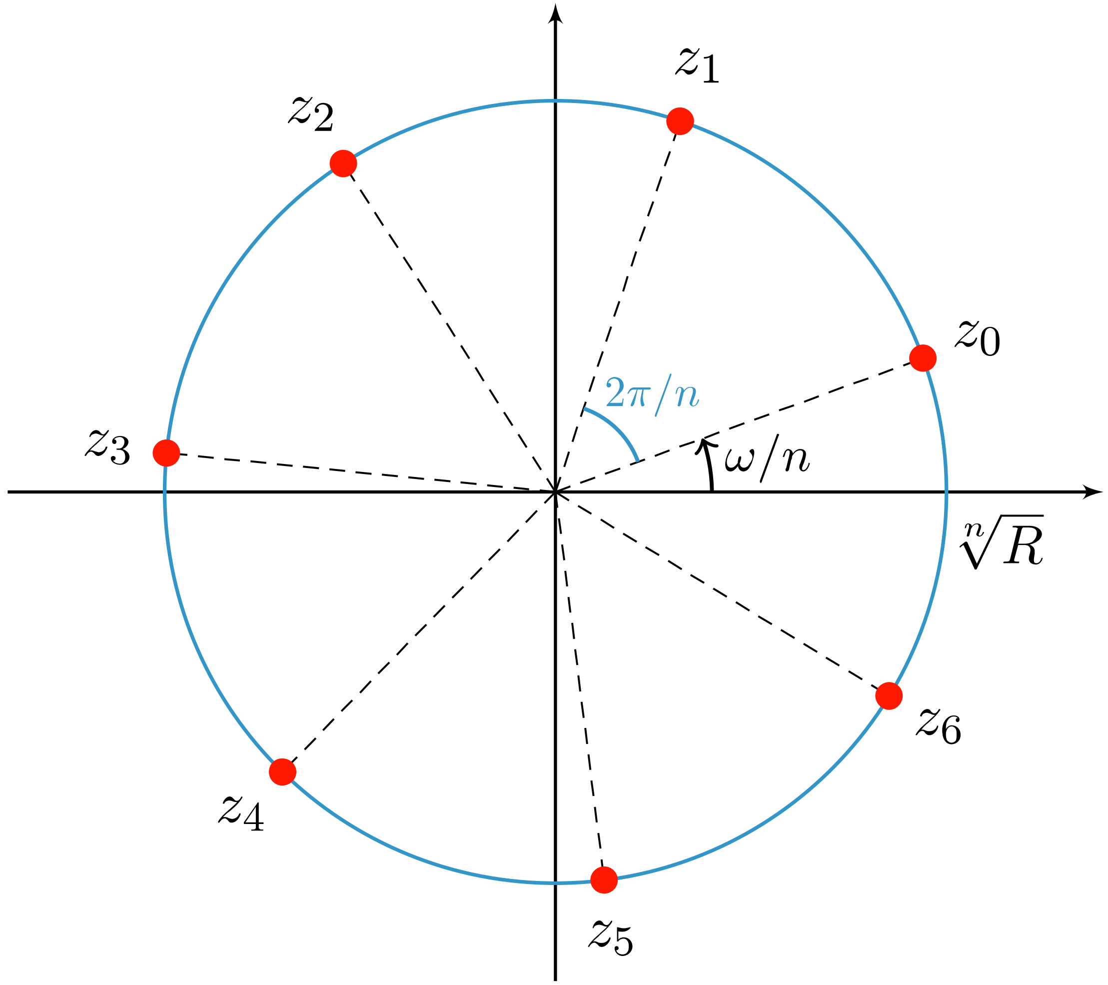
$$\boxed{z_k = \sqrt[n]{R}\,e^{i\left(\frac{\omega}{n} + \frac{2k\pi}{n}\right)}, \quad k = 0, 1, \ldots, n-1}$$

Finding roots

The solutions of $z^n = u = R\,e^{i\omega}$.



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Find all complex numbers z such that $z^4 = 16$.

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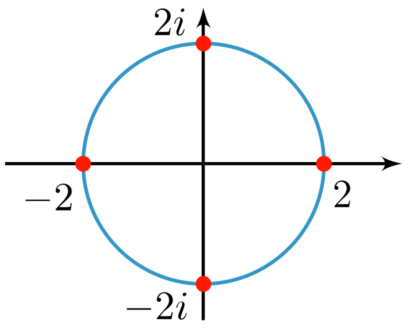
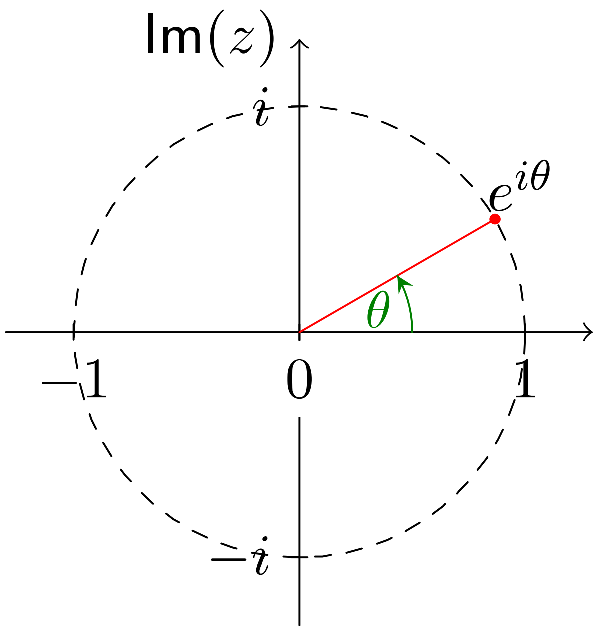
- The solutions of $z^4 = 16$ are

$$z_1 = 2e^{i0} = 2, \qquad z_2 = 2e^{i\pi/2} = 2i,$$
$$z_3 = 2e^{i\pi} = -2, \quad z_4 = 2e^{i3\pi/2} = -2i.$$



The complex exponential as a function

We can regard $e^{i\theta}$ as a function depending on θ.



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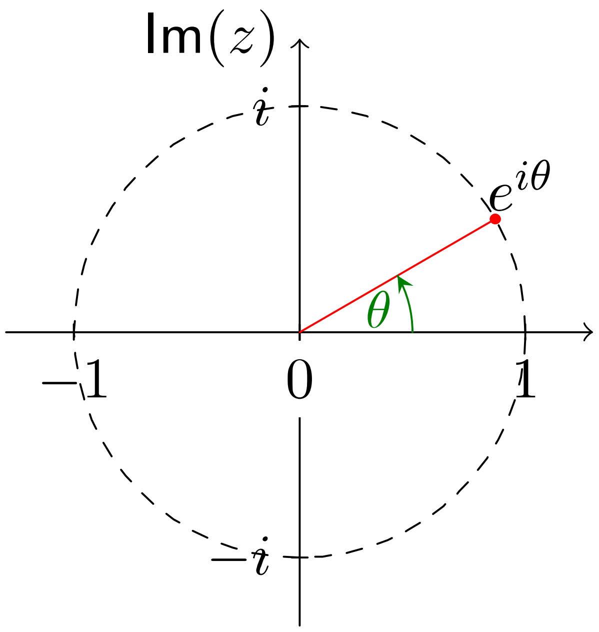

This function has a derivative $\cdots$

The complex exponential as a function

- We calculate the derivative of $e^{i\theta}$ as follows:

$$\frac{d\,e^{i\theta}}{d\theta} = \left(e^{i\theta}\right)' = (\cos\theta + i\sin\theta)'$$

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- Similarly, for real values of r:

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- Similarly, for real values of r:

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- Note that for real values r we have

$$\left(e^{r\theta}\right)' = re^{r\theta}.$$

Summarizing Exercise

Define

$$z = \frac{4}{1+\sqrt{3}i}$$

(a) Find the real and imaginary part of z
(b) Find the absolute value of z
(c) Find the argument of z
(d) Find the smallest positive number n such that z^n is real

Find all solutions of the equation

$$\frac{1}{z-i} = \overline{z} + i$$

The complex exponential as a function

Definition

For $z = a + bi$, we define the exponential as

$$e^{a+bi} = e^a \cdot e^{bi} = e^a \left[\cos(b) + i\sin(b)\right]$$

Theorem

Let $z = a + bi \in \mathbb{C}$, *then*

$$\boxed{\frac{d\, e^{z\theta}}{d\theta} = \left(e^{z\theta}\right)' = z e^{z\theta}.}$$

-Contents-

- ❑ √ Integrals
- ❑ √ Calculation techniques for integrals √
- ❑ Power and Taylor series
- ❑ √ First order ODEs
- ❑ √ Complex numbers
- ❑ Second order ODEs

https://www.youtube.com/watch?v=3Uo0JAUWijM