Lecture 4 – First order differential equations

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Differential equations – what are they?

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A solution is a function that, whenever substituted, satisfies the equation.

- The function $y(x) = e^x - x - 1$ satisfies (1):

  $$y'(x) = e^x - 1,$$

  consequently

  $$y'(x) - y(x) = (e^x - 1) - (e^x - x - 1)$$

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consequently

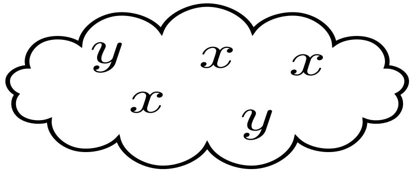
$$\begin{aligned} y'(x) - y(x) &= (\cancel{e^x} - \cancel{1}) - (\cancel{e^x} - x - \cancel{1}) \\ &= x. \end{aligned}$$

Differential equations – first-order equations

Definition

An ordinary first-order differential equation is a differential equation where y' can be expressed as a function of y and x.

$$y' =$$



- The formal way to denote

$$y' = f(x, y).$$

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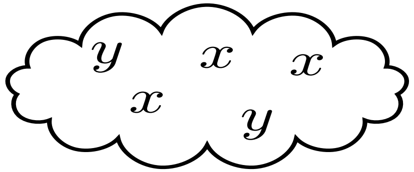
$y' =$ 

- The formal way to denote

$$y' = f(x, y).$$

Example

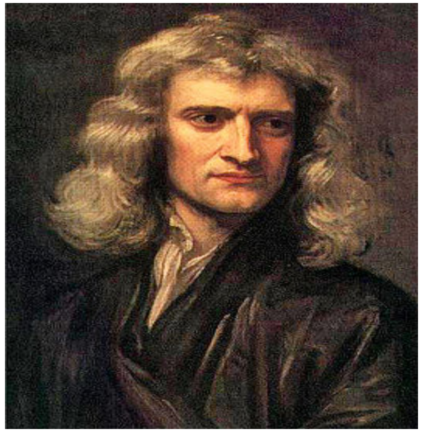
The equation $y' - y = x$ can be rewritten as a normal equation:

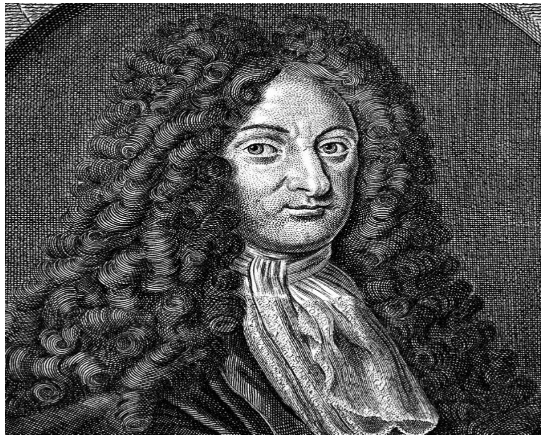
$$y' = y + x. \qquad (2)$$

Hence we have: $f(x, y) = y + x$.

Differential equations – the making of

Differential equations first appeared in the work of Newton and Leibniz, ± 1672.





They were first constructed to model physics, but are nowadays used to model almost everything, stock prices, glucose concentration in the body, traffic flows, etc.

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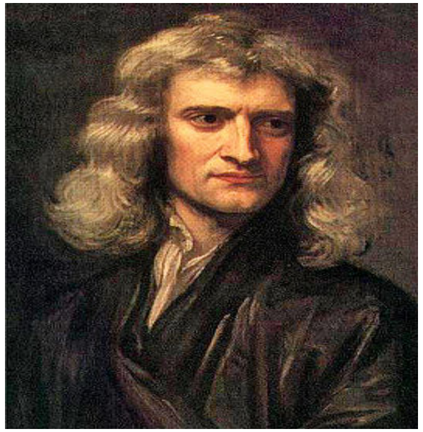
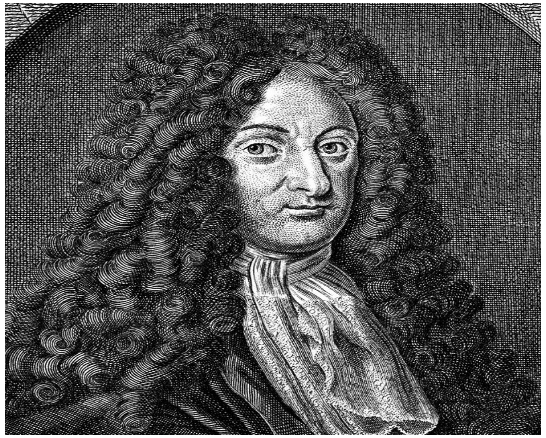
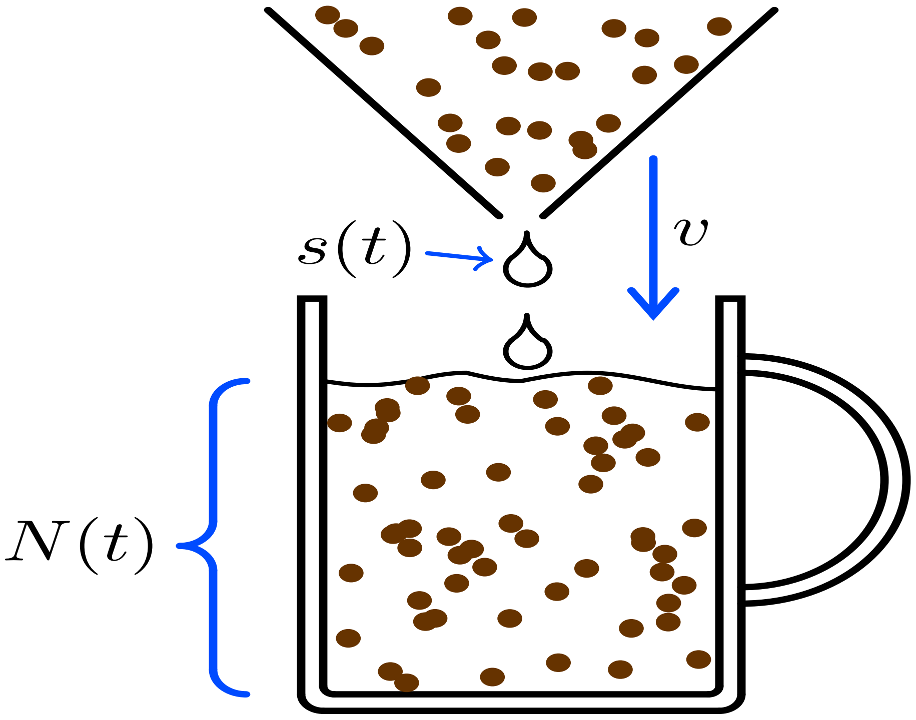




They were first constructed to model physics, but are nowadays used to model almost everything, stock prices, glucose concentration in the body, traffic flows, etc.
Next we derive a differential equation modelling the making of a mug of coffee via filtration.

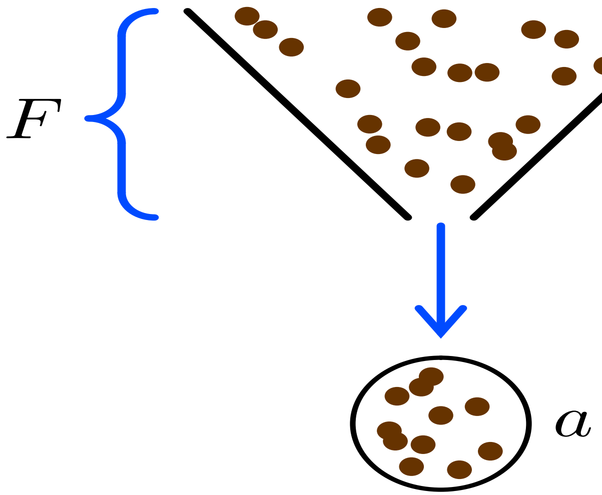
Differential equations – making coffee



Const	description	units
F_0	initial number of coffee particles in the filter	–
k	coffee filtration constant	s^{-1}
v	flow through filter	m^3/s
T_m	time needed to fill one mug	s
T_s	moment that the mugs have to be switched	s

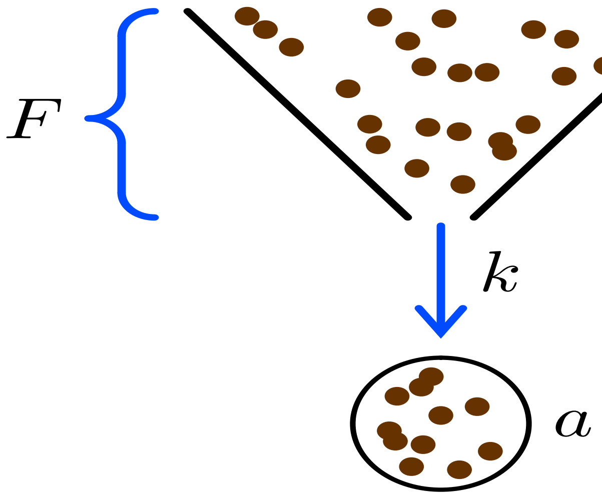
Var	description	units
t	time	s
$N(t)$	nr of coffee particles in mug	–
$s(t)$	coffee strength at time t	m^{-3}
α	strength decay ratio for one mug: $\alpha = \frac{s(T_m)}{s(0)}$	–

Differential equations – making coffee



- The number a of coffee particles that leaves the filter during a short period of time Δt is proportional to the number of coffee particles F in the filter and to the length of the period Δt:

$$a \propto F\Delta t.$$



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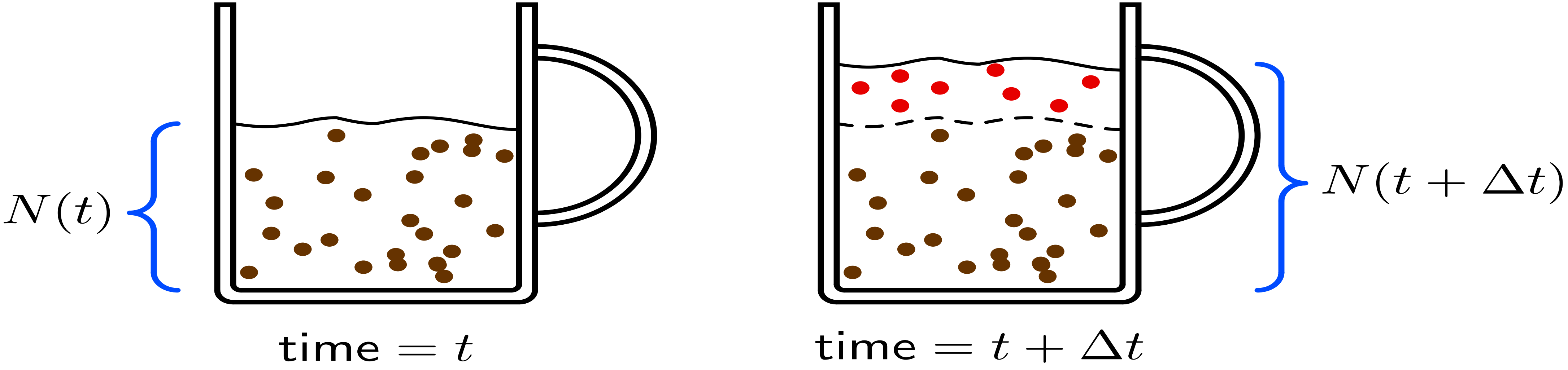
- The coffee constant k is defined as the fraction particles that trickle through the filter per second, hence

$$a = kF\Delta t.$$

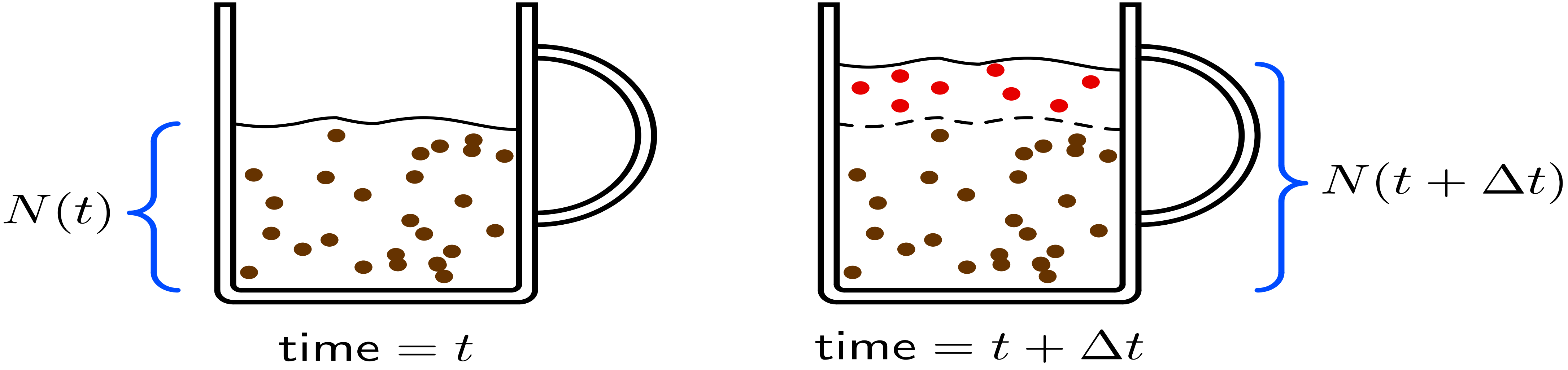
Differential equations – making coffee



- The number of coffee particles that enters the mug between time t and $t + \Delta t$ is

$$N(t + \Delta t) - N(t).$$

Differential equations – making coffee

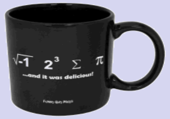


- The number of coffee particles that enters the mug between time t and $t + \Delta t$ is

$$N(t + \Delta t) - N(t).$$

- The number of coffee particles that leaves the filter during a period of Δt seconds is approximately

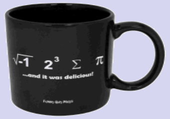
$$k\big(F_0 - N(t)\big)\Delta t.$$



- The function N satisfies the differential equation

$$\boxed{N' = k\big(F_0 - N\big)}$$

Differential equations – making coffee



- No coffee is wasted or added, so

$$N(t+\Delta t) - N(t) \approx k\big(F_0 - N(t)\big)\Delta t.$$

- Divide left and right hand side by Δt:

$$\frac{N(t+\Delta t) - N(t)}{\Delta t} \approx k\big(F_0 - N(t)\big).$$

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- The approximation becomes better as Δt gets smaller:

$$N'(t) = \lim_{\Delta t \to 0} \frac{N(t+\Delta t) - N(t)}{\Delta t} = k\big(F_0 - N(t)\big).$$

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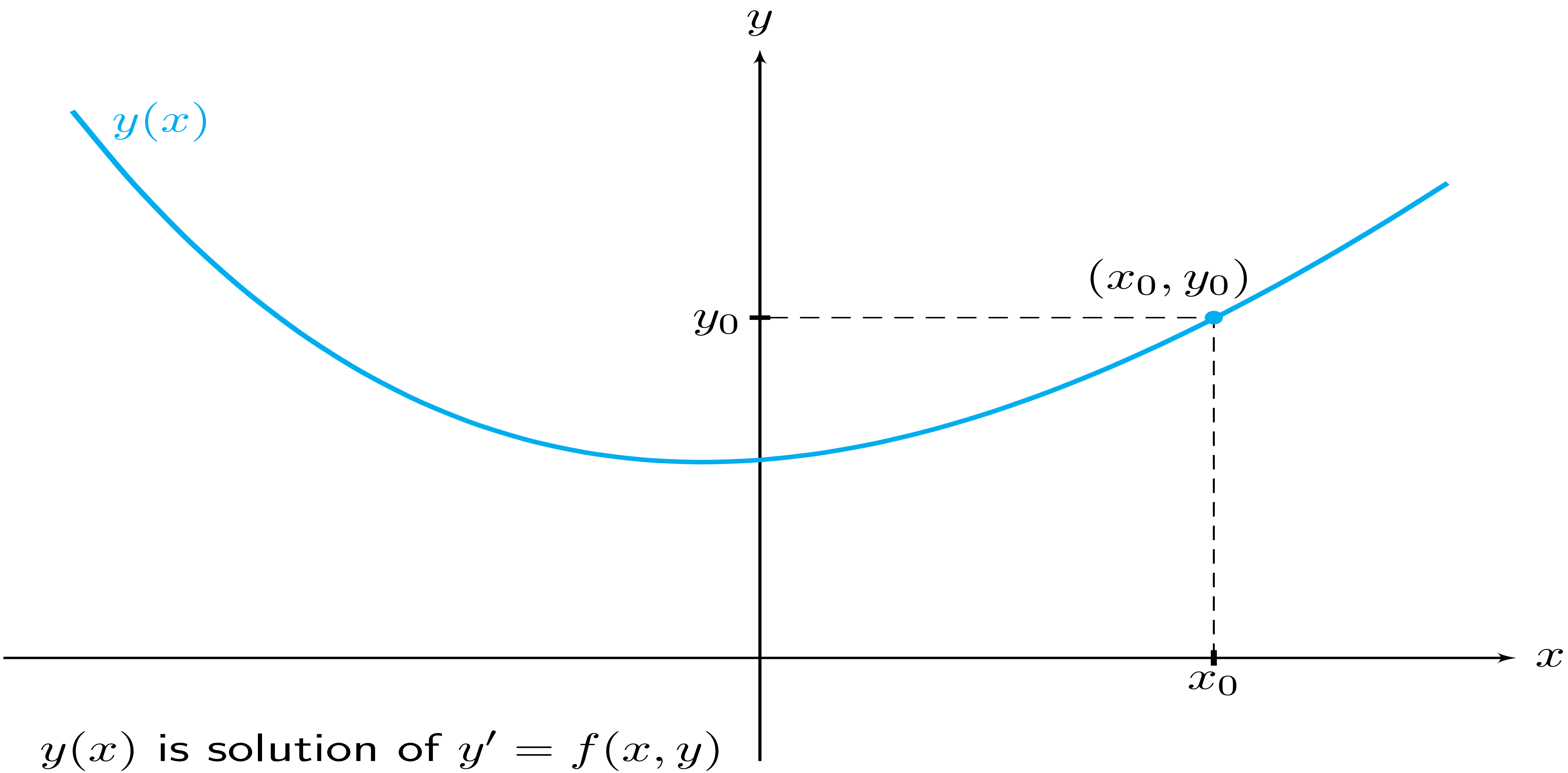
Differential equations – solutions

Given a differential equation, a natural question is how to find a solution.
We will show two techniques:

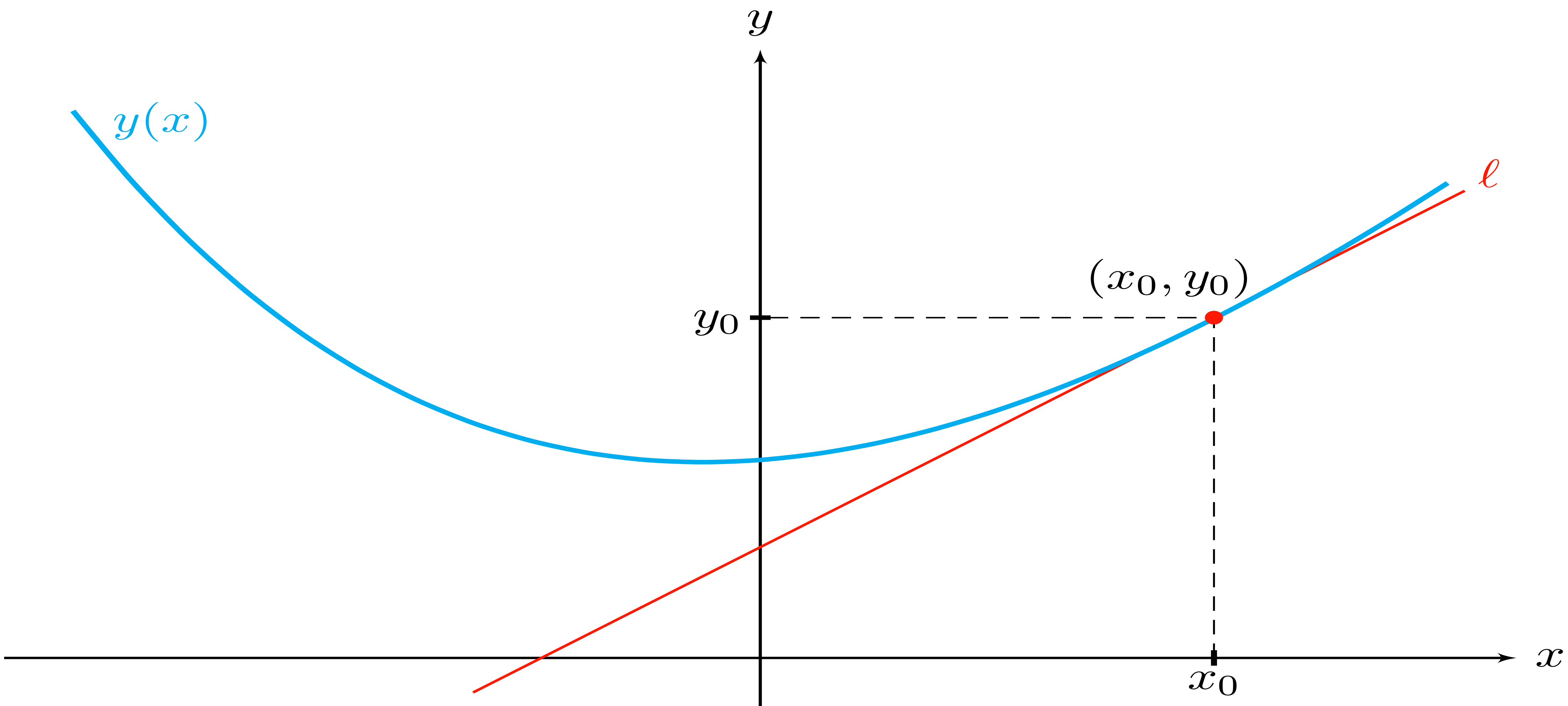
- Direction fields, which is graphical construction;
- Integrating factor, which is a way for calculating the solution.

Differential equations – direction fields



$y(x)$ is solution of $y' = f(x, y)$
and $y_0 = y(x_0)$

Differential equations – direction fields



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slope of ℓ is $y'(x_0) = f(x_0, y_0)$

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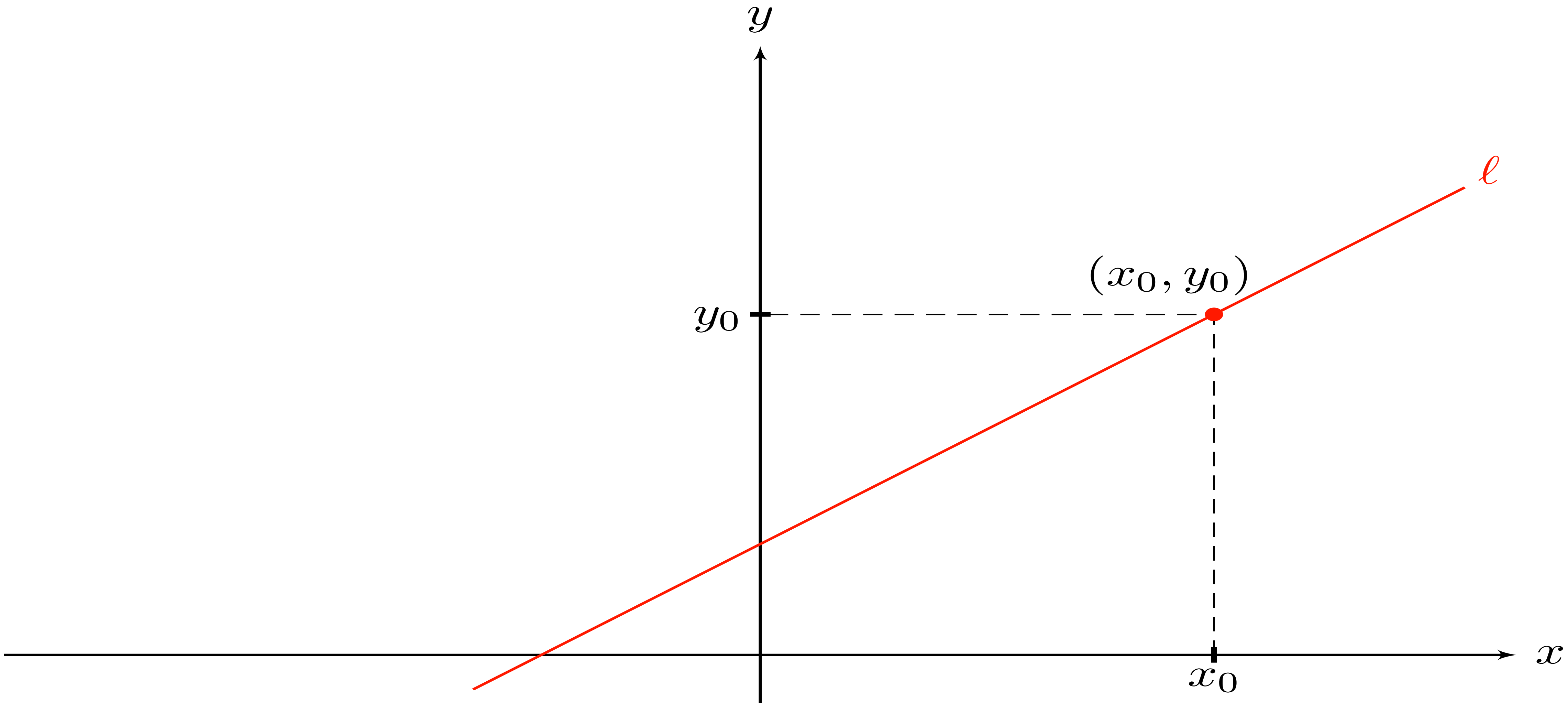

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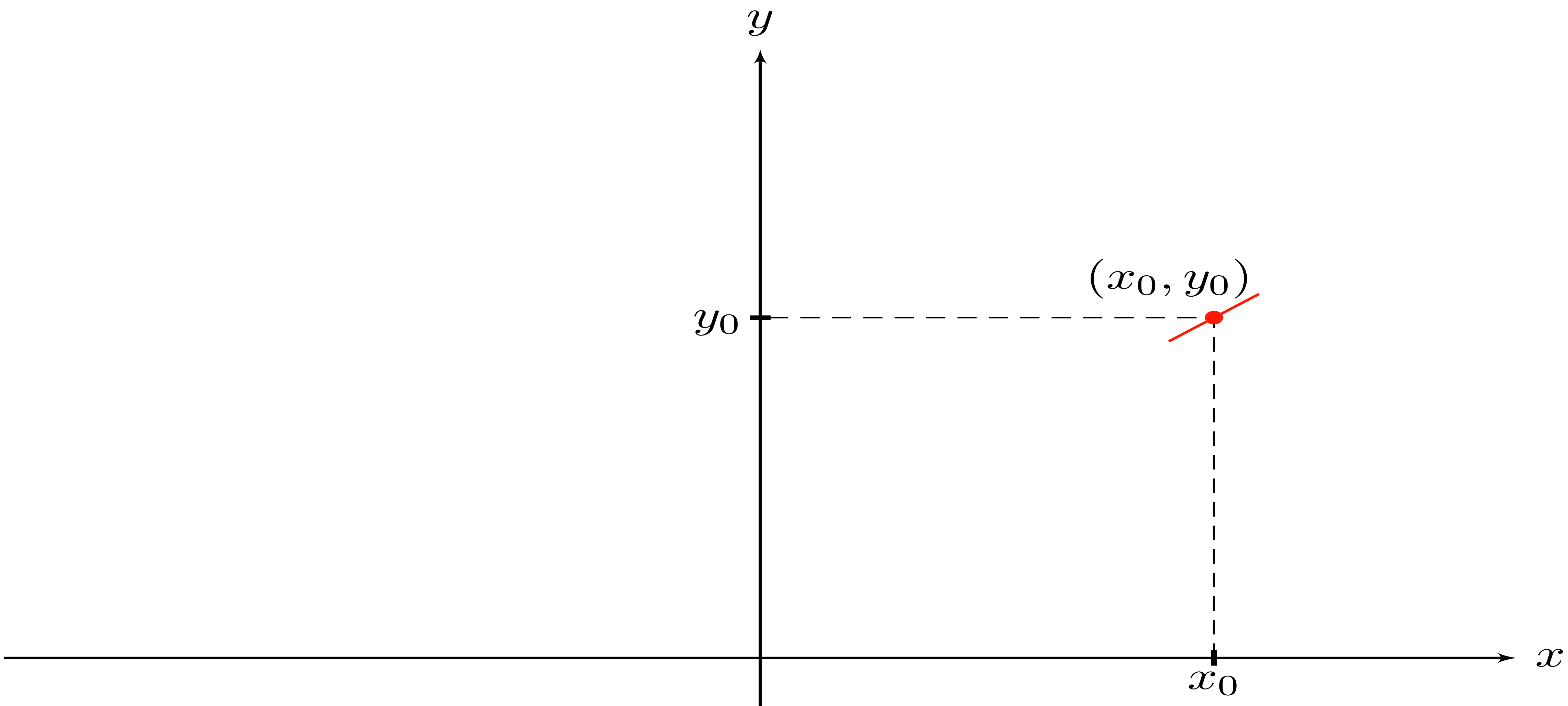

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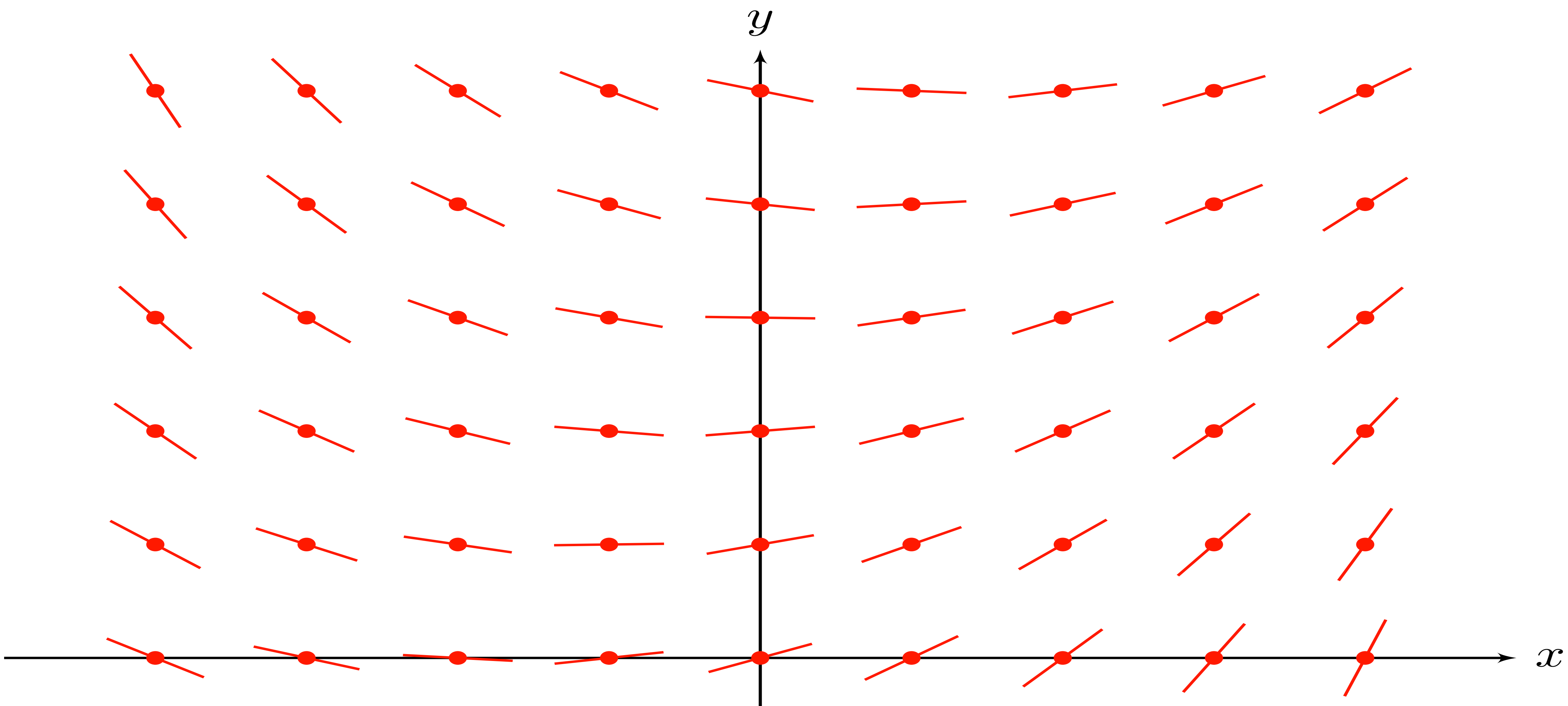

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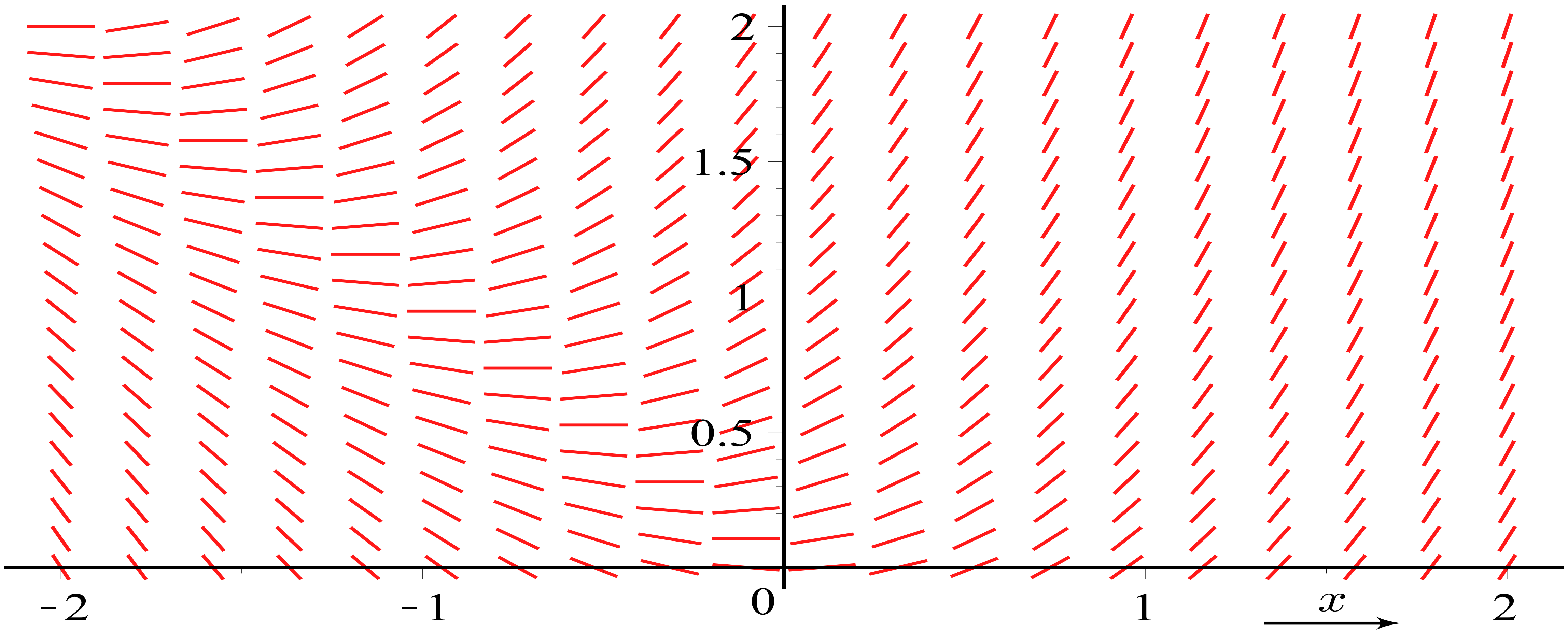
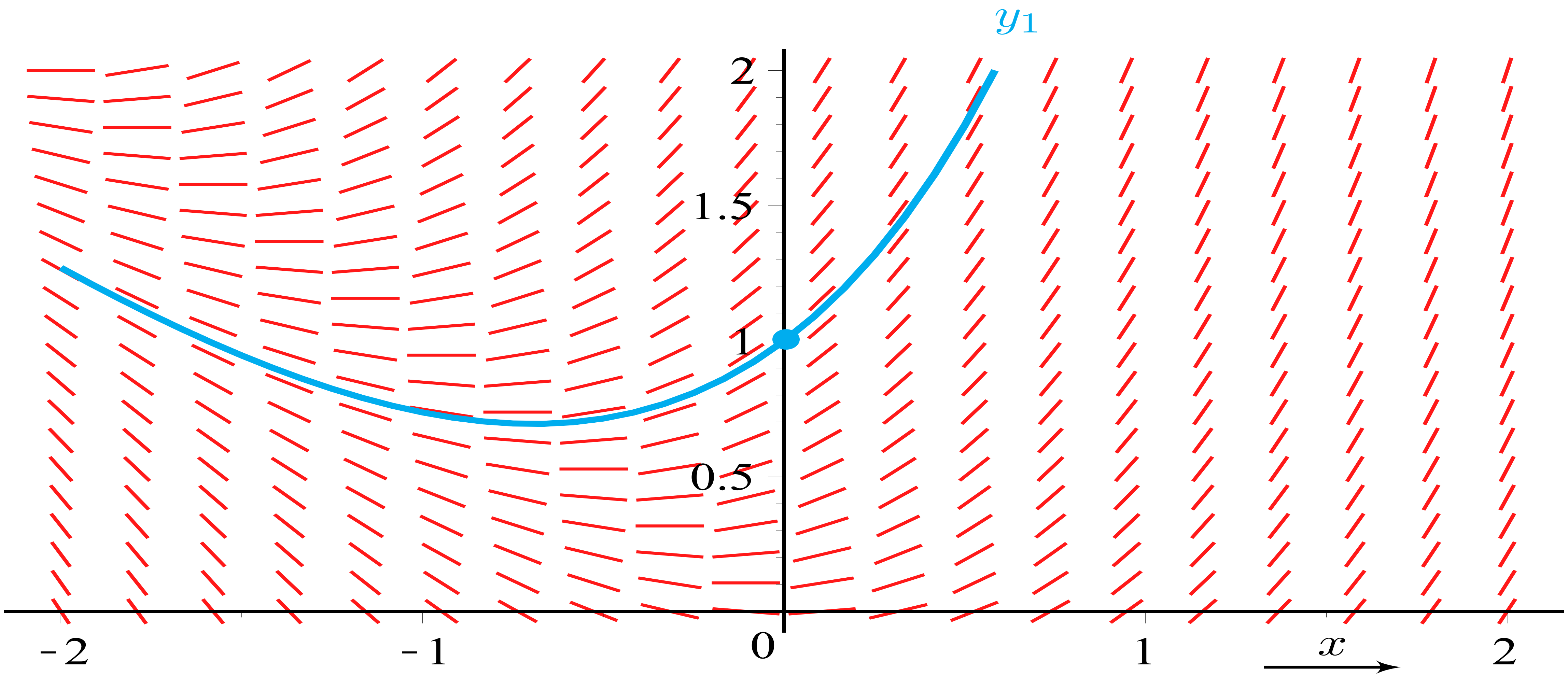


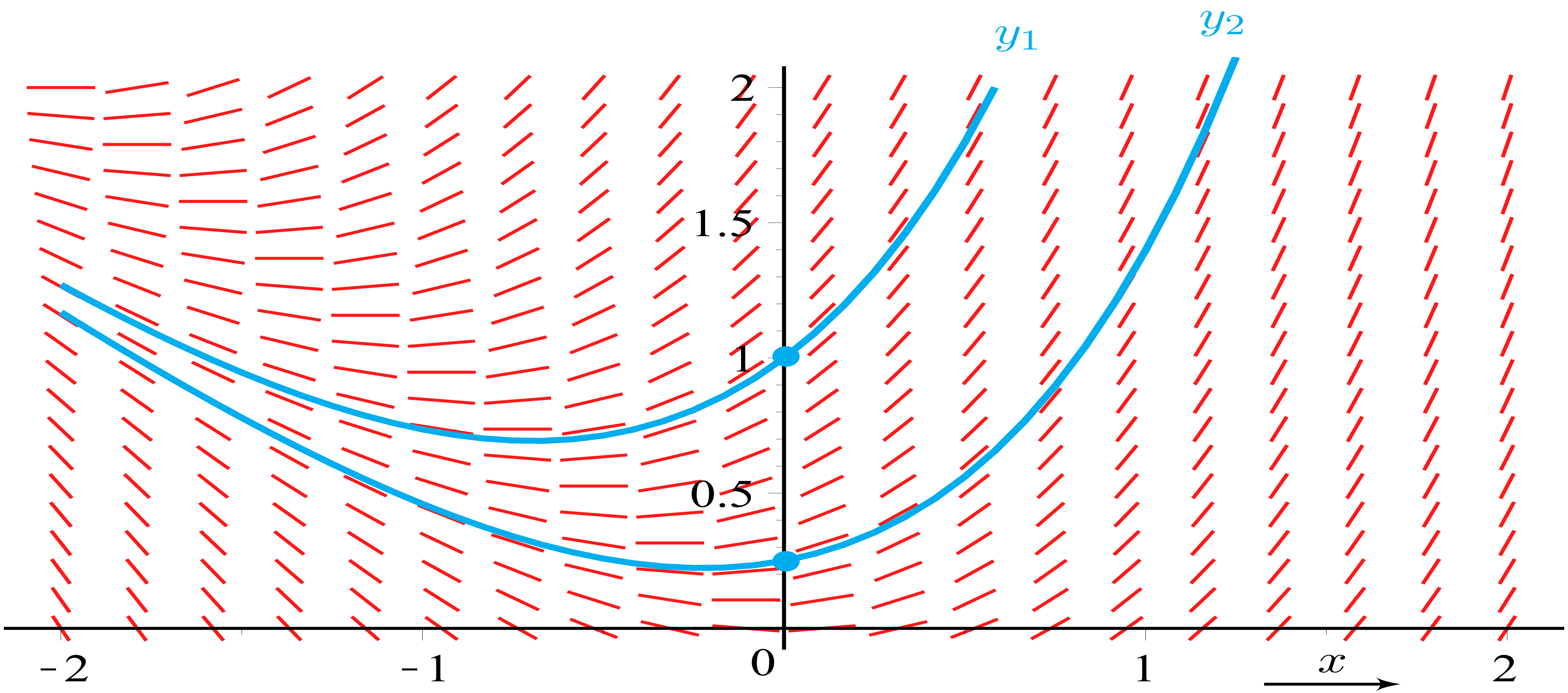
$y' = y + x$

Differential equations – direction fields



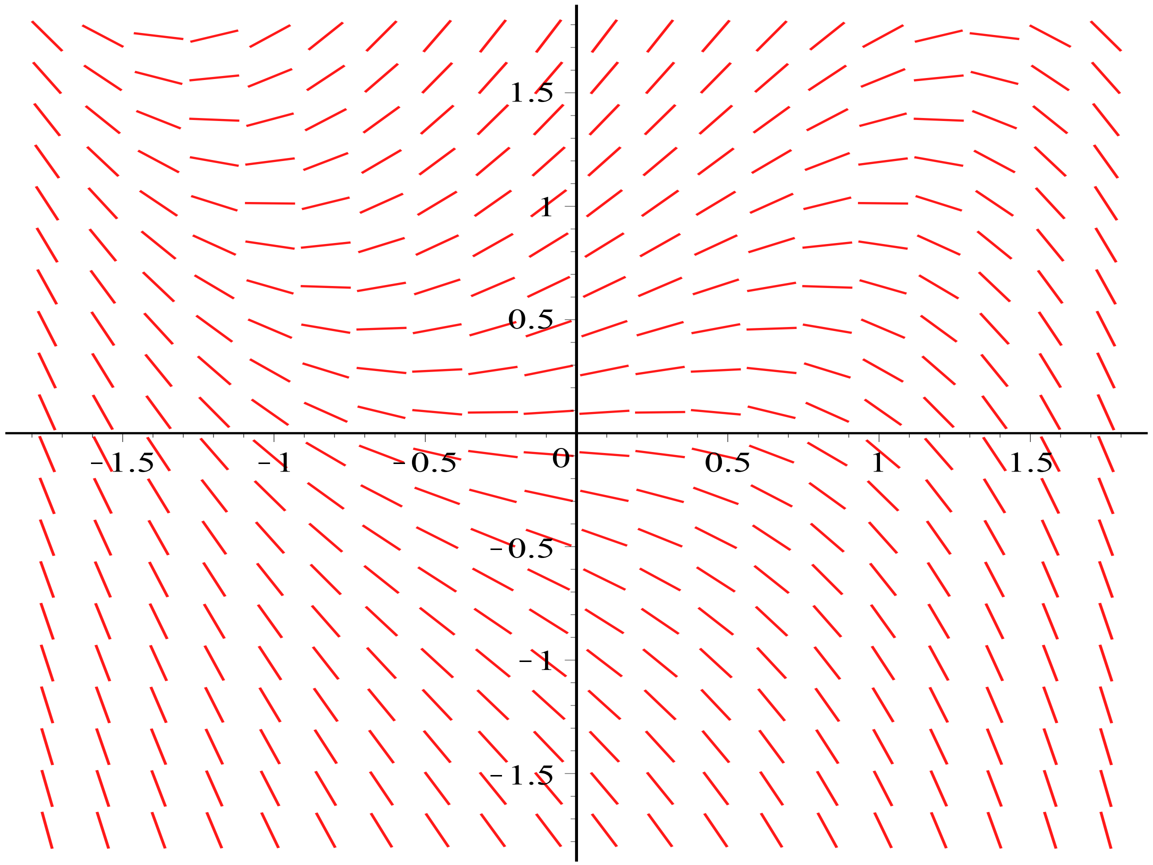
$y' = y + x$ $\qquad$ $y_1(0) = 1$

Differential equations – direction fields

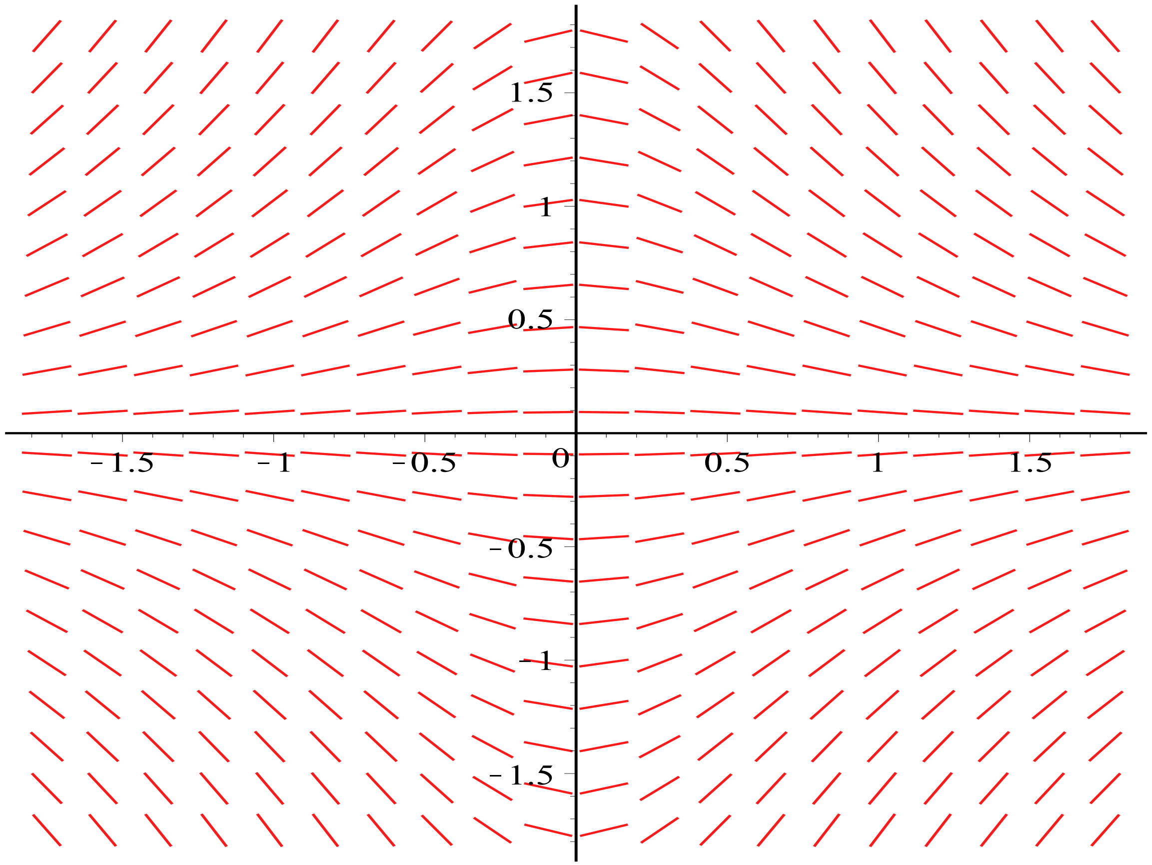


$y' = y + x$ $\qquad$ $y_1(0) = 1$ $\qquad$ $y_2(0) = .25$

Differential equations – direction fields

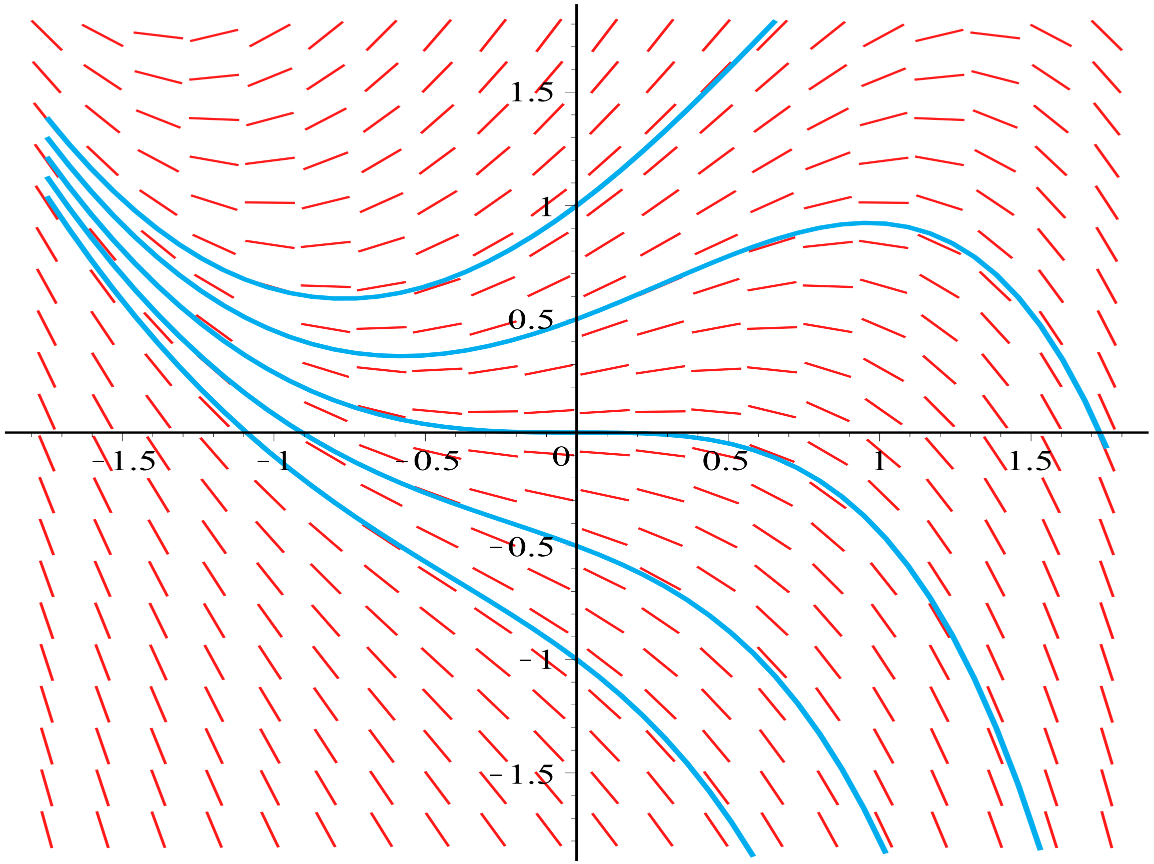


$$y' = y - x^2$$



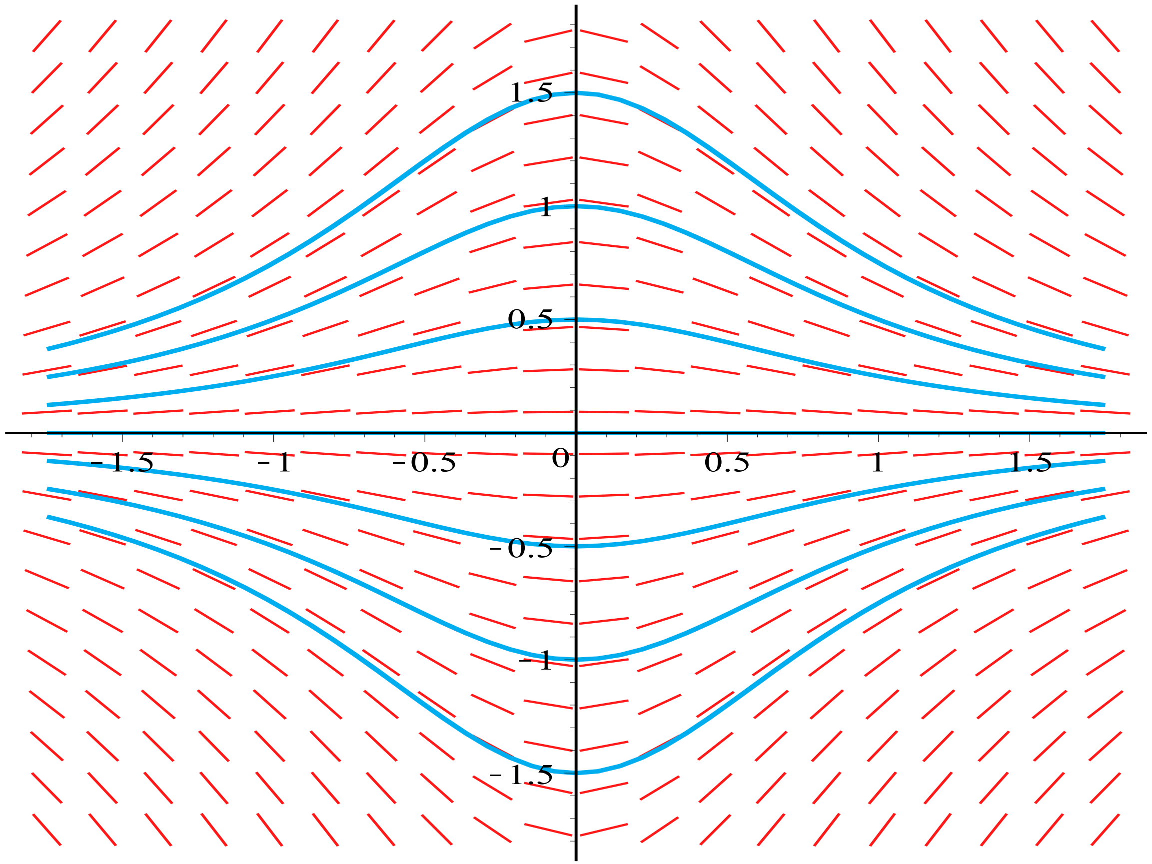
$$y' = -\frac{2xy}{x^2+1}$$

Differential equations – direction fields



$$y' = y - x^2$$

$$y(x) = x^2 + 2x + 2 + Ce^x$$



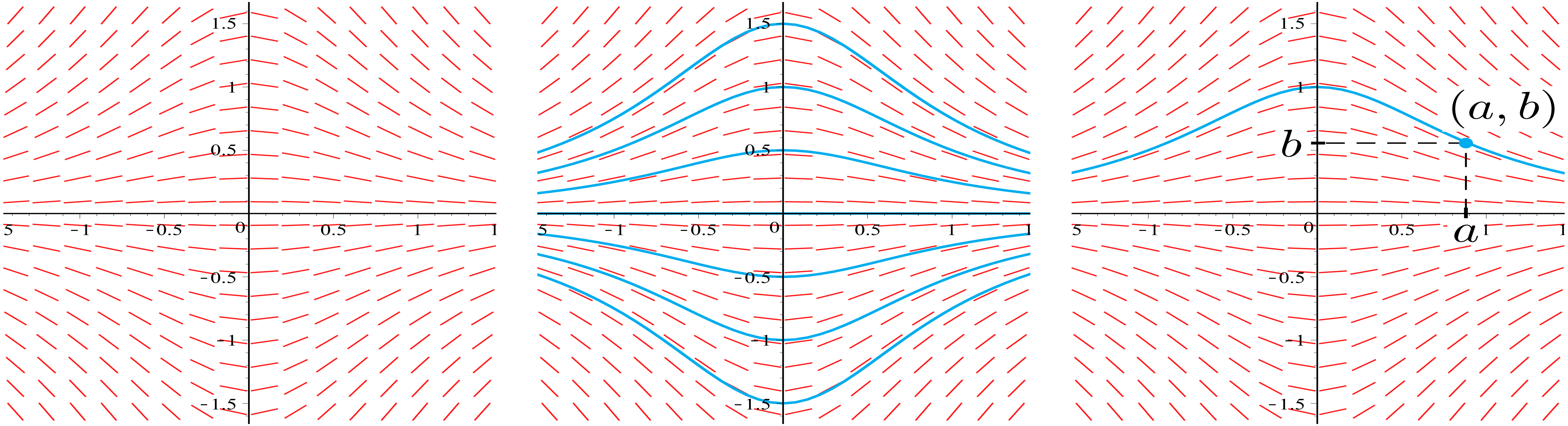
$$y' = -\frac{2xy}{x^2+1} \qquad y(x) = \frac{C}{x^2+1}$$

- In general a differential equation has infinitely many solutions.

Definition

An initial value problem consists of a differential equation and an equation of the form $y(a) = b$.

- An initial value problem in general has one solution.



Differential equations – exponential growth

Definition

A quantity y depending on time grows exponentially if $y' \propto y$.

- If y grows exponentially, then y satisfies the differential equation

$$y' = ky \qquad (1)$$

for some constant k.

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Quiz A solution of equation (1) is given by

(a) $y(x) = k\,e^{x}$; (b) $y(x) = -5e^{kx}$;

(c) $y(x) = 7e^{-kx}$; (d) $y(x) = \ln(kx)$;

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(c) $y(x) = 7e^{-kx}$;
(d) $y(x) = \ln(kx)$;

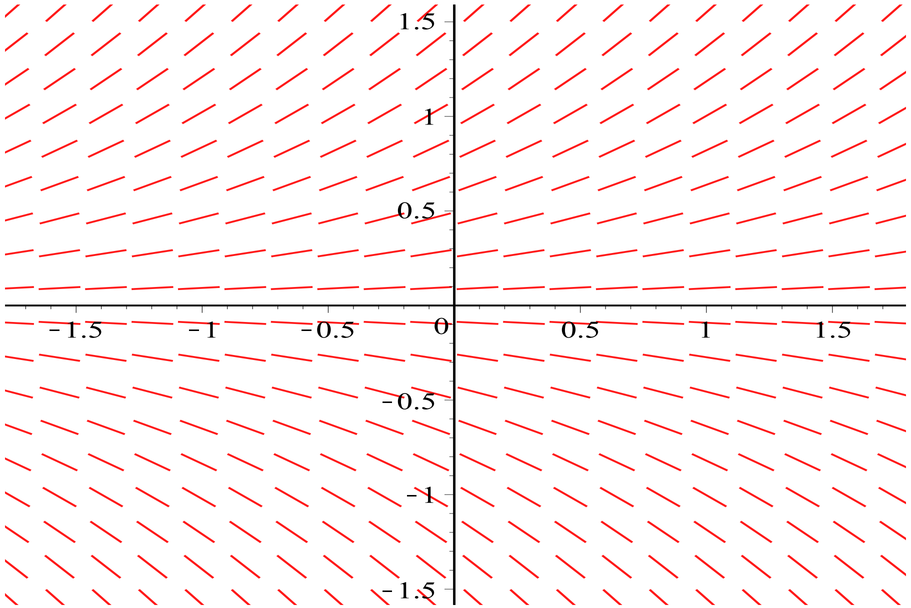
$(b) \quad y(x) = -5e^{kx} \quad \rightarrow \quad y'(x) = -5ke^{kx} = k(-5e^{kx}) = ky(x)$

Differential equations – exponential growth

Examples of exponential grow are:

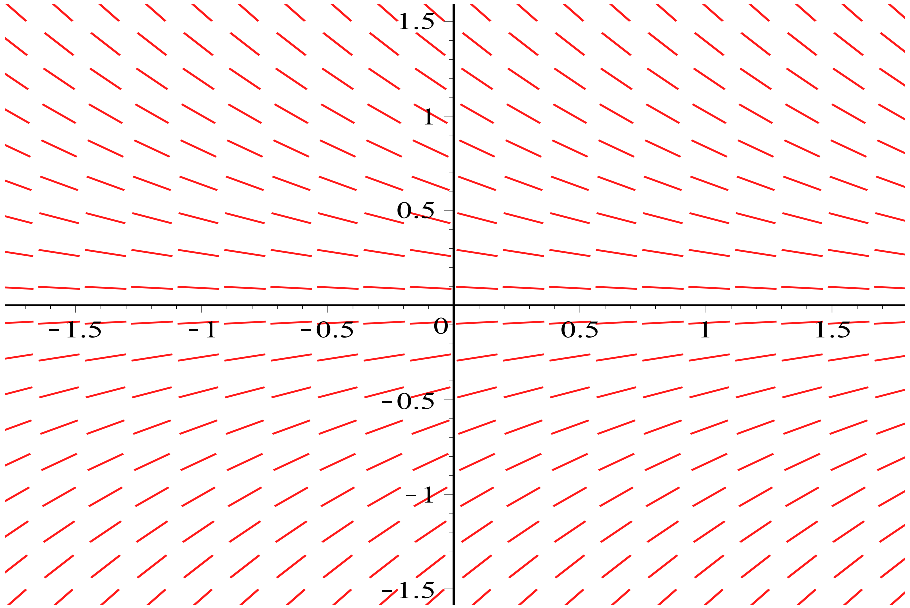
- the human population (annual growth percentage is approximately 1.14%);
- duckweed growth in a pond;
- carbon dating (the half-life of ^{14}C is approximately 5730 years);
- compound interest;
- Moore's law: the number of transistors on integrated circuits doubles approximately every two years.

Differential equations – exponential growth



$k > 0$

$$\boxed{y' = ky}$$

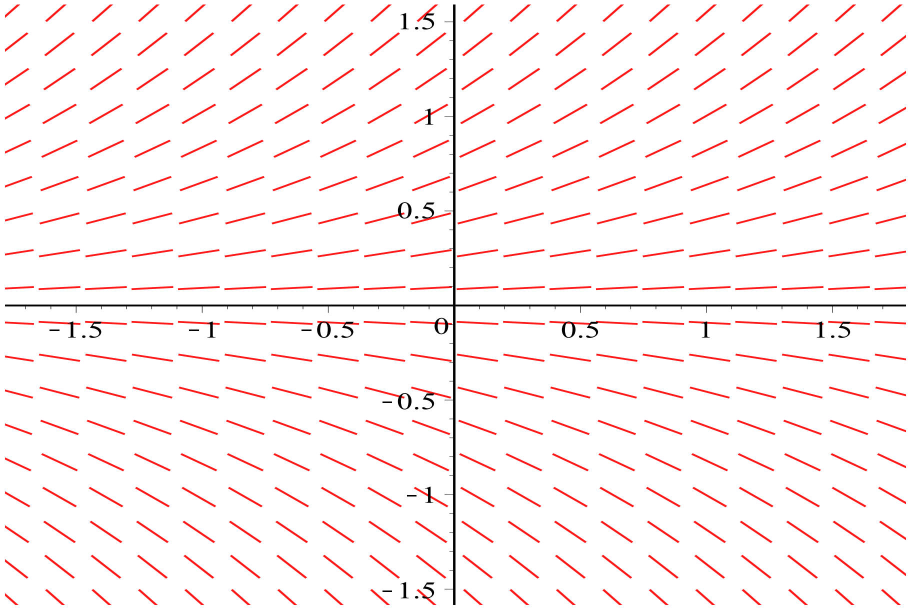


$k < 0$

- Let $y(t)$ be a solution. Define $Y(t) = y(t)e^{-kt}$, then

$$Y' = y'e^{-kt} - kye^{-kt}$$

Differential equations – exponential growth



$k > 0$

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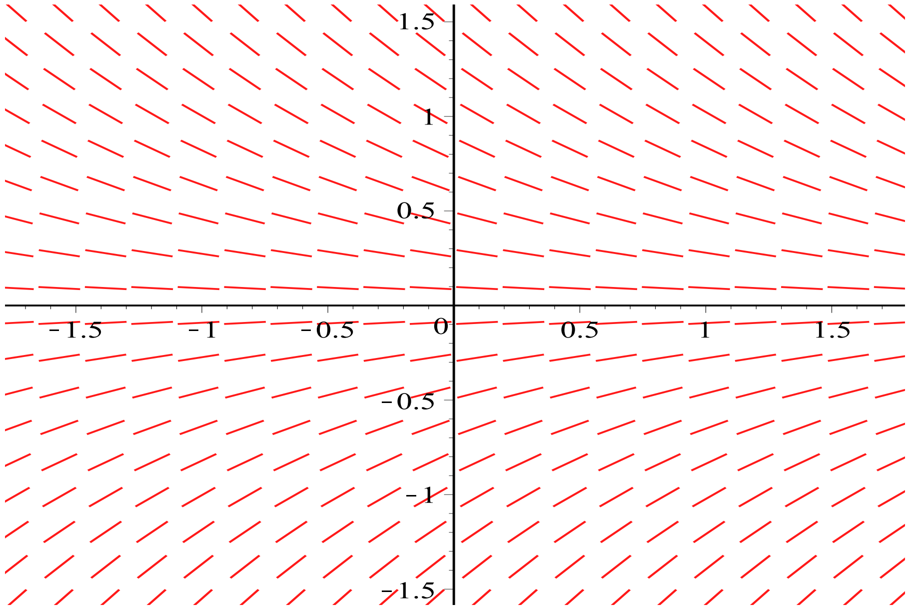


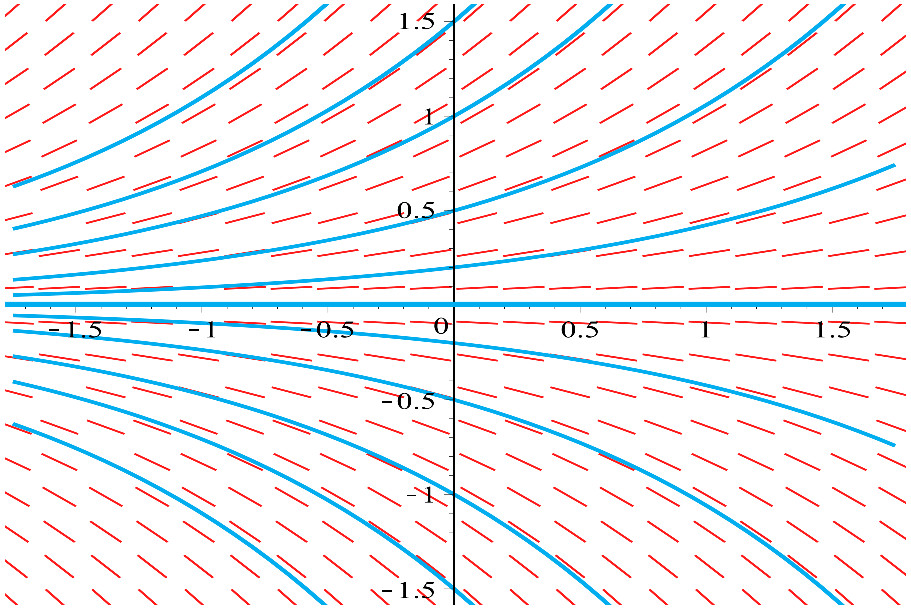
$k < 0$

- Let $y(t)$ be a solution. Define $Y(t) = y(t)e^{-kt}$, then

$$\begin{aligned} Y' &= y'e^{-kt} - kye^{-kt} \\ &= (y' - ky)e^{-kt} = 0, \end{aligned}$$

hence $Y(t)$ is constant.

Differential equations – exponential growth



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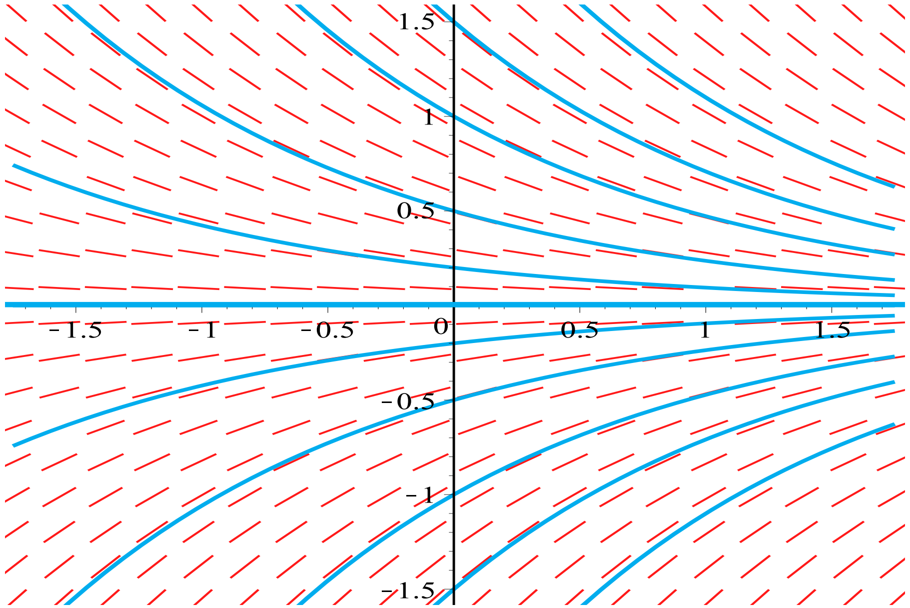

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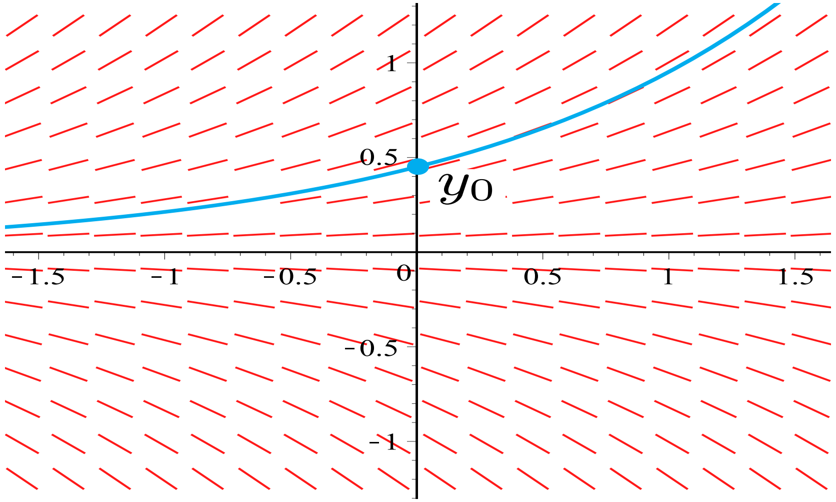
$$\begin{aligned} Y' &= y'e^{-kt} - kye^{-kt} \\ &= (y' - ky)e^{-kt} = 0, \end{aligned}$$

  hence $Y(t)$ is constant.
- The general solution of $y' = ky$ is $y(t) = Ce^{kt}$ with $C \in \mathbb{R}$.

Differential equations – initial value solution



Exponential growth with an initial condition

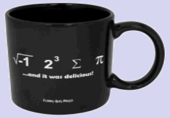
The (unique) solution of the initial value problem

$$\begin{cases} y' = ky, \\ y(0) = y_0 \end{cases}$$

is $y(t) = y_0 e^{kt}$.

- We often write $y(t) = y(0)e^{kt}$.

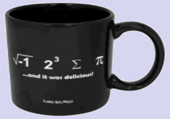
Differential equations – making coffee



- The coffee problem is described by the initial value problem

$$\begin{cases} N' = k(F_0 - N) \\ N(0) = 0. \end{cases}$$

Differential equations – making coffee



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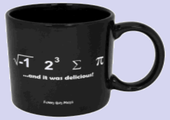
- The number of coffee particles in the filter at time t is described by

$$F(t) = F_0 - N(t).$$

Differential equations – making coffee



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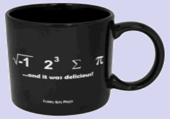
$$F(t) = F_0 - N(t).$$

- The quantity F is exponentially decaying:

$$\begin{cases} F' = -N' = -k(F_0 - N) = -kF \\ F(0) = F_0 - N(0) = F_0, \end{cases}$$

therefore $F(t) = F_0 e^{-kt}$.

Differential equations – making coffee



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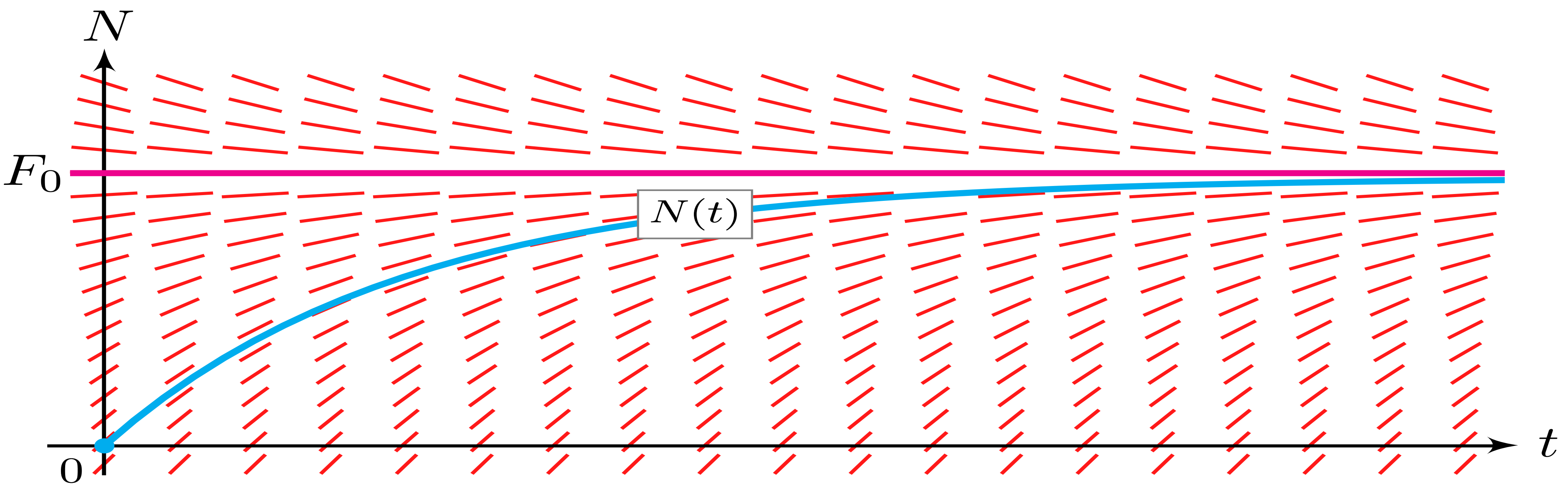
- $$\boxed{N(t) = F_0\left(1 - e^{-kt}\right)}$$

Differential equations – making coffee



- The number of coffee particles in the mug at time t is

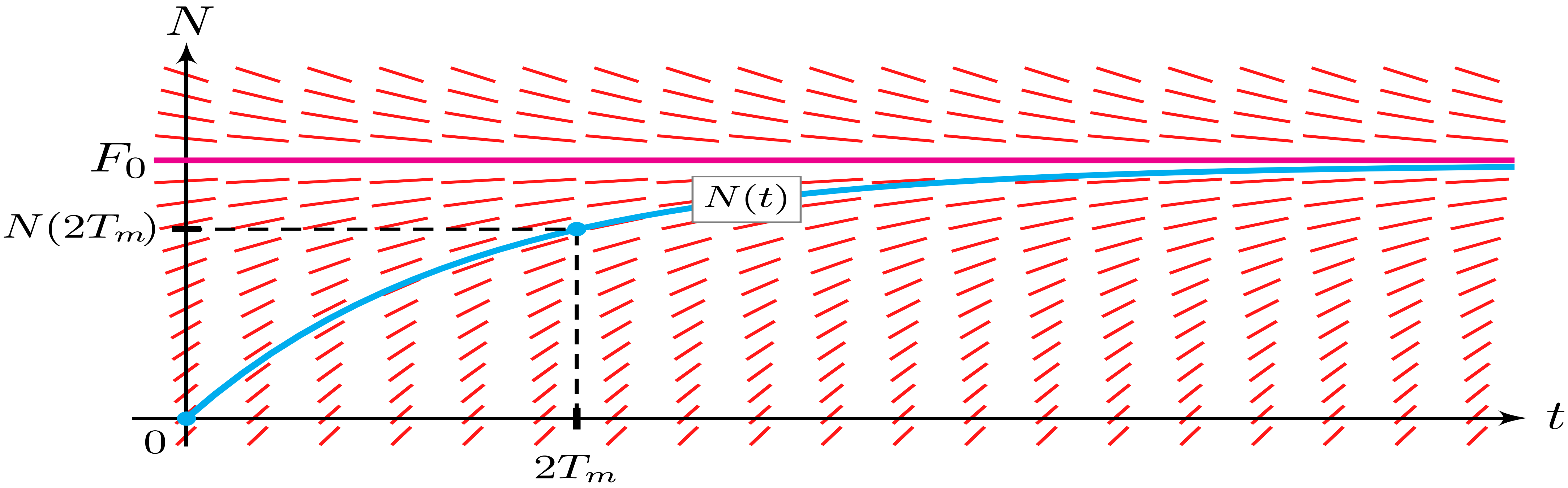
$$N(t) = F_0\left(1 - e^{-kt}\right), \qquad t \geq 0.$$

- For large t almost all coffee particles will be extracted from the filter into the brew.

Coffee – a modeling problem

http://www.amazon.com

Differential equations – making coffee



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- For large t almost all coffee particles will be extracted from the filter into the brew.
- Usually there will be coffee particles left after filling two mugs.

Rule of 70

Let y be a quantity that grows exponentially with a growth percentage $r\%$ per time unit. If r is small, the doubling time is approximately $\frac{70}{r}$ time units.

- If duckweed grows with with 1.5% per hour, then the total biomass is doubled in $\frac{70}{1.5} \approx 46$ hours, which is about 2 days. Starting with a patch of $1\,\mathrm{m}^2$, a pond with a surface area of $128\,\mathrm{m}^2$ will be covered with duckweed after approximately two weeks.

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- If $v' = vP$ then $(vy)' = vQ$.
  Consequently vy (and therefore y) can be obtained by integrating vQ.

Differential equations – linear first-order equations

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v is called *integrating factor*

Differential equations – integrating factor

Solve the differential equation $xy' - y = x^2\,e^x$.

write in standard form	$y' + Py = Q$	
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divide by v	$y = \frac{1}{v}\int v(x)Q(x)\,\mathrm{d}x$	$y(x) = x\left(e^x + C\right)$

Summarizing Exercise

Solve

$$xy' + \frac{y}{x} = 0 \qquad ; \qquad y(1) = 1$$

Find particular solution of

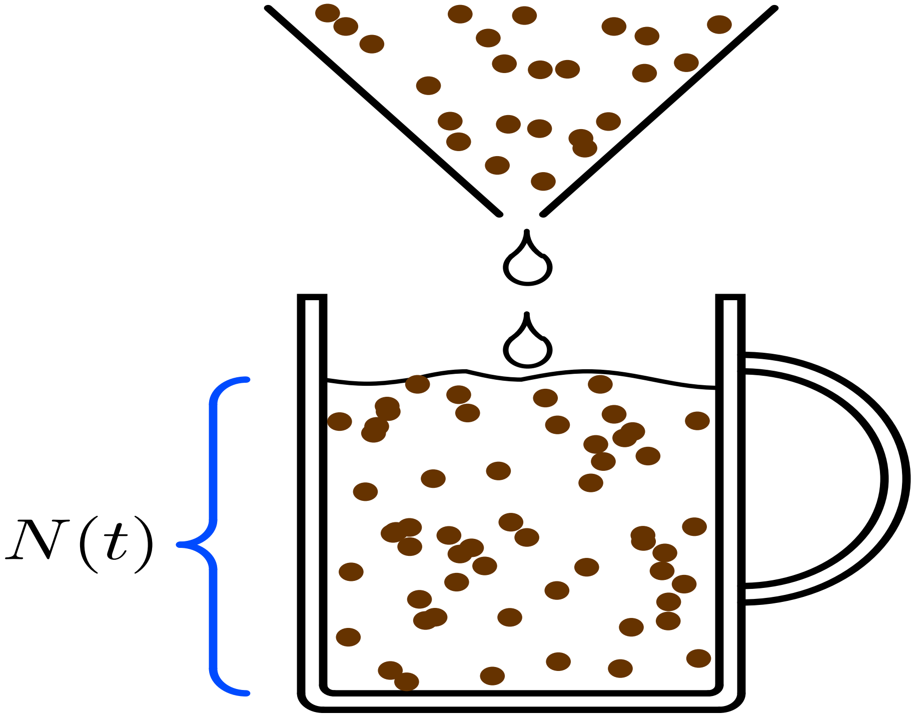
$$xy' + \frac{y}{x} = e^{1/x}$$

-Contents-

- [x] Integrals
- [x] Calculation techniques for integrals
- [x] Power and Taylor series

- [x] First order ODEs
- [ ] Complex numbers
- [ ] Second order ODEs

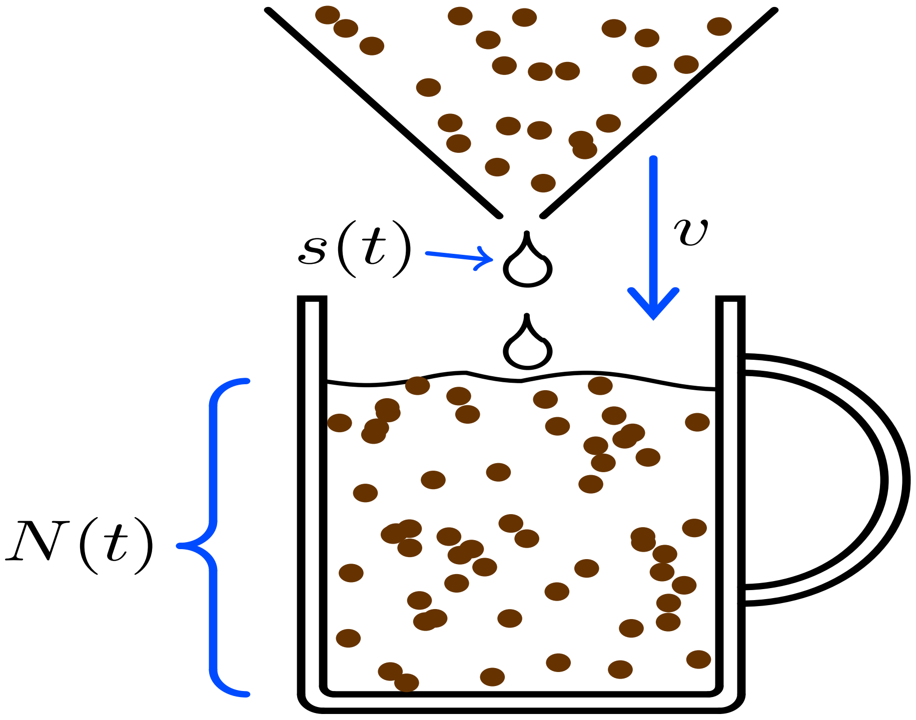
Differential equations – making coffee



As model for the strength we have the differential equation

$$N' = k(F_0 - N).$$

We want to use this model to decide when to switch the mugs, when we want to make two mugs of coffee with the same content (volume and strength).



Const	description	units
F_0	initial number of coffee particles in the filter	–
k	coffee filtration constant	s^{-1}
v	flow through filter	m^3/s
T_m	time needed to fill one mug	s
T_s	moment that the mugs have to be switched	s

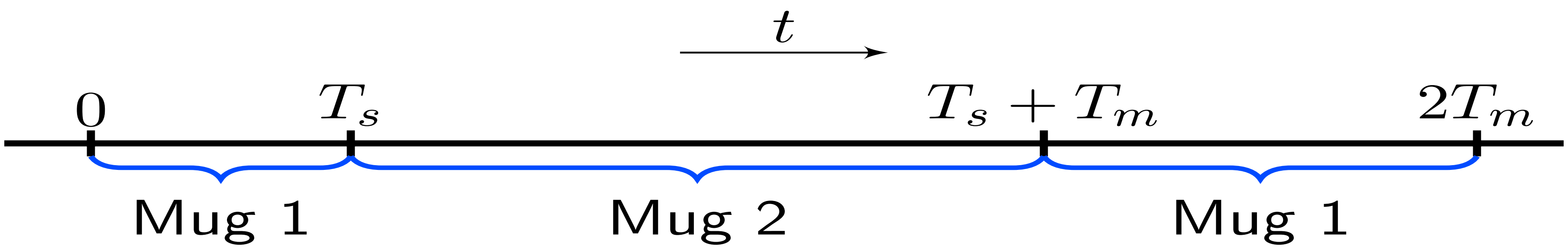
Var	description	units
t	time	s
$N(t)$	nr of coffee particles in mug	–
$s(t)$	coffee strength at time t	m^{-3}
α	strength decay ratio for one mug: $\alpha = \dfrac{s(T_m)}{s(0)}$	–

Differential equations – making coffee

Solving strategy

- Let T_s be the moment that the mugs are interchanged, then

$$\underbrace{N(T_s)}_{\substack{\text{strong coffee}\\ \text{in mug 1}}} + \underbrace{N(2T_m) - N(T_s + T_m)}_{\text{weak coffee in mug 1}} = \underbrace{N(T_s + T_m) - N(T_s)}_{\text{coffee in mug 2}}\,.$$



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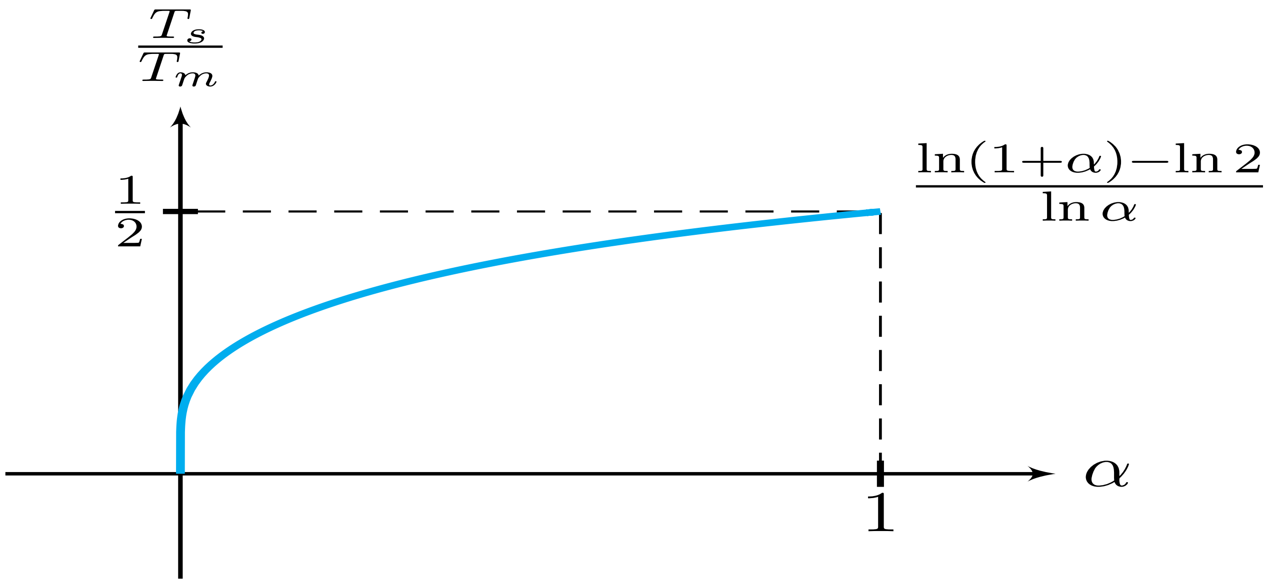

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$$\frac{T_s}{T_m} = \frac{\ln(1+\alpha) - \ln 2}{\ln \alpha}.$$

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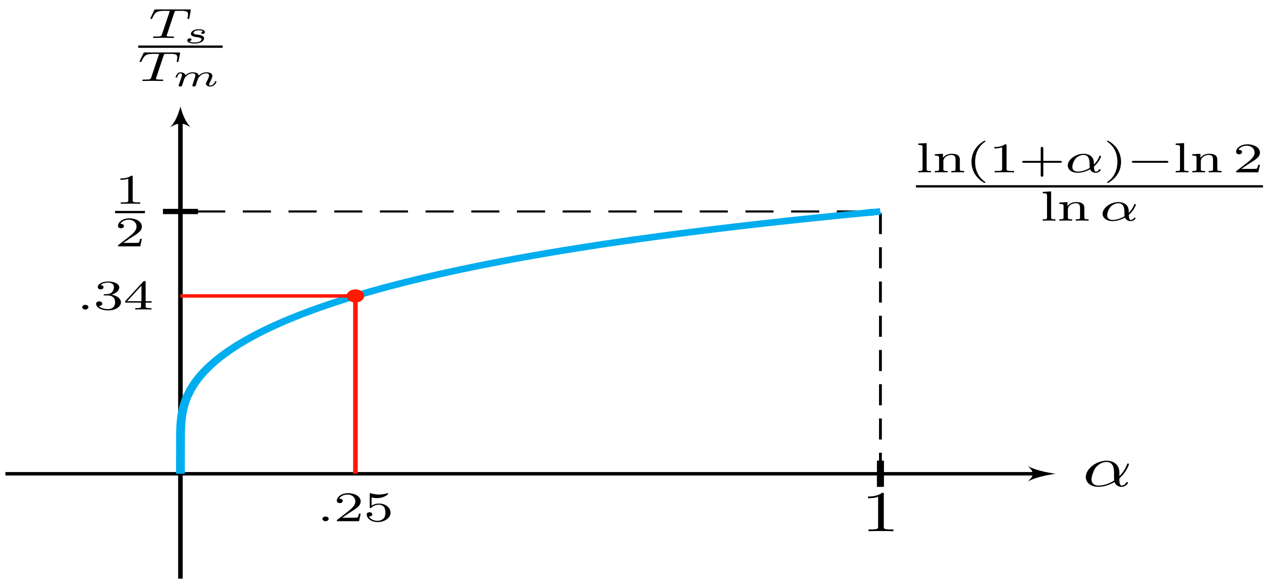

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- Example: if the coffee is initially 4 times as strong as the coffee at time T_m, then $\alpha = .25$ and $\frac{T_s}{T_m} \approx 0.34$.

  The mugs should be exchanged if the first mug is 34% filled.