Mathematics B2: Newton

Brook Taylor (1685-1731)

Mathematics B2: Newton

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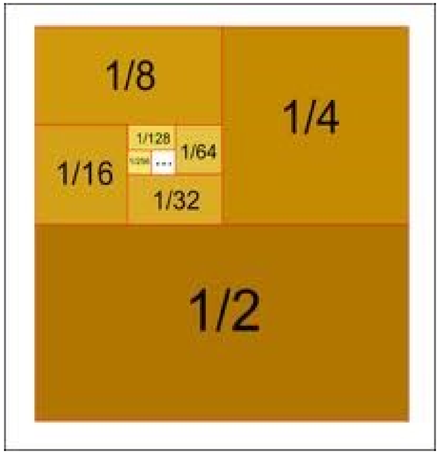
INFINITE SEQUENCES AND SERIES



$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1$$

SUM OF INFINITE SERIES

Given a series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

SUM OF INFINITE SERIES

If the sequence $\{s_n\}$ is convergent and

$\lim_{n\to\infty} s_n = s$ exists as a real number, then the series

$\Sigma\, a_n$ is called ***convergent*** and we write:

$$a_1 + a_2 + \cdots + a_n + \cdots = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

- The number s is called the ***sum*** of the series.

- Otherwise, the series is called ***divergent***.

SUM OF INFINITE SERIES vs IMPROPER INTEGRALS

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

Compare with the improper integral

$$\int^{\infty} f(x)\,dx = \lim_{t\to\infty} \int^{t} f(x)\,dx$$

- To find this integral, we integrate from 1 to t and then let $t \to \infty$.

GEOMETRIC SERIES

GEOMETRIC SERIES

An important example of an infinite series is the geometric series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots$$

$$= \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

If $r = 1$, then

$$s_n = a + a + \cdots + a = na \to \pm\infty \text{ (divergent)}$$

GEOMETRIC SERIES

If $r \neq 1$, we have

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

and

$$rs_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

Subtracting these equations, we get:

$$s_n - rs_n = a - ar^n$$

So: $$s_n = \frac{a(1 - r^n)}{1 - r}$$

GEOMETRIC SERIES

$$s_n = \frac{a(1 - r^n)}{1 - r}$$

- If $-1 < r < 1$, we know that $r^n \to 0$ as $n \to \infty$, so

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{a}{1 - r} \lim_{n \to \infty} r^n = \frac{a}{1 - r}$$

- If $r \le -1$ or $r > 1$, the sequence $\{r^n\}$ is divergent

Theorem:

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$. The sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \geq 1$, the series is divergent.

Example 1

$$a + ar + ar^2 + ar^3 + \cdots$$

Find the sum *s* of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

- The first term is $a = 5$ and the common ratio is $r = -2/3$.

Hence $s = \dfrac{a}{1-r} = \dfrac{5}{1-(-2/3)} = \dfrac{5}{5/3} = 3$

Example 2

Is the series $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ convergent or divergent?

Example 2 (solution)

Let's rewrite the *n*th term of the series in the form ar^{n-1}:

$$\sum_{n=1}^{\infty} 2^{2n} 3^{1-n} = \sum_{n=1}^{\infty} (2^2)^n 3^{-(n-1)} = \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}$$

- We recognize this series as a geometric series with $a = 4$ and $r = 4/3$.
- Since $|r| > 1$, the series diverges!

$$s = \frac{a}{1 - r}$$

Example 3

Write the number $2.3\overline{17} = 2.3171717\ldots$ as a ratio of two integers.

- $2.3171717\ldots = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$
- After the first term (2.3), we have a geometric series with a = 17/1000 and r = 1/100.

Therefore: $2.3\overline{17} = 2.3 + \frac{17/1000}{1 - 1/100} = \ldots = \frac{1147}{495}$

Example 4

Find the sum of the series $\sum_{n=0}^{\infty} x^n$ where $|x| < 1$.

- Notice that this series starts with $n = 0$.
- So, the first term is $x^0 = 1$.
- We adopt the convention that $x^0 = 1$ even when $x = 0$.

Example 4 (continued)

Thus,

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

is a geometric series with $a = 1$ and $r = x$.

Since $|r| = |x| < 1$, it converges, and

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

POWER SERIES

Definition:

A ***power series*** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

where:

- x is a variable.
- The c_n's are constants called the coefficients of the series.

Definition (continued):

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

The ***sum*** of the above power series is the function

$$f(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots$$

whose domain is the set of all x for which the series converges.

Example:

If we take $c_n = 1$ for all n, the power series is the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots + x^n + \ldots$$

which converges when $-1 < x < 1$ and diverges when $|x| \geq 1$.

Definition (continued):

More generally, a series of the form

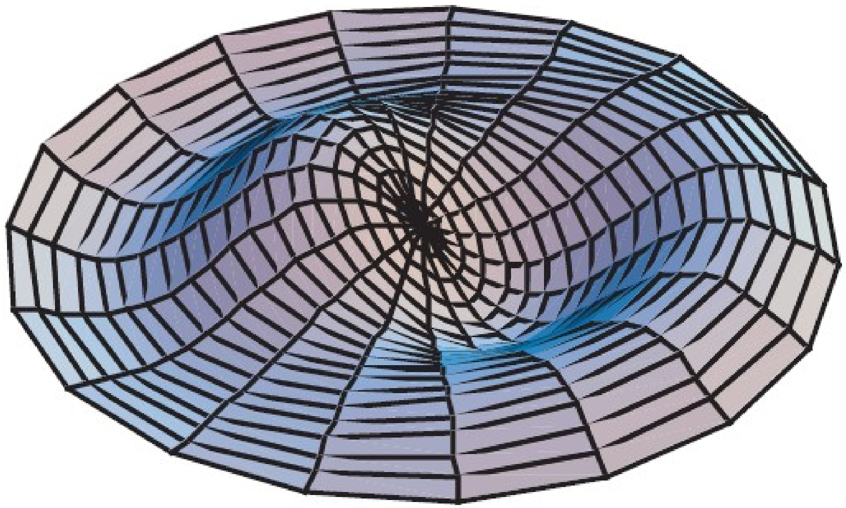
$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

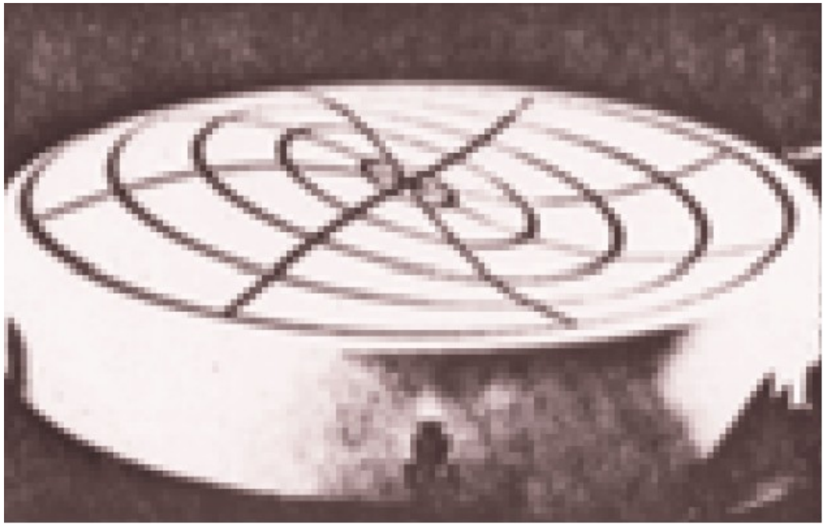
is called any of the following:

- A power series in $(x - a)$
- A power series centered at a
- A power series about a

BESSEL FUNCTION

Notice how closely the computer-generated model (which involves Bessel functions and cosine functions) matches the photograph of a vibrating rubber membrane.





The sum of the power series below is called a ***Bessel function***, after the German astronomer Friedrich Bessel (1784–1846).

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

RADIUS OF CONVERGENCE

POWER SERIES

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

In power series the set of values of x for which the series is convergent is always an interval:

- A finite interval for the geometric series, case $c_n = 1$
- The infinite interval $(-\infty, \infty)$ for the Bessel series
- A collapsed interval $[0, 0] = \{0\}$ in case $c_n = n!$

Remark:

Note that the inequality $|x - a| < R$ can be rewritten as $a - R < x < a + R$.

When x is an endpoint of the interval, that is, $x = a \pm R$, anything can happen:

- The series might converge at one or both endpoints.
- It might diverge at both endpoints.

Definition:

The ***interval of convergence*** of a power series

$$\sum_{n=0}^{\infty} c_n (x - a)^n$$

is the interval around a that consists of all values of x for which the series converges.

The number R is called the *radius* of convergence of the power series.

Theorem:

For a given power series $\sum_{n=0}^{\infty} c_n (x - a)^n$

around a there are only three possibilities:

I. The series converges only when $x = a$, denoted by $R = 0$.

II. The series converges for all x , denoted by $R = \infty$.

III. There is a positive number R such that the series *converges* for $|x - a| < R$ and *diverges* for $|x - a| > R$.

GEOMETRIC SERIES

GEOMETRIC SERIES

We start with an expression we have seen before; we'll refer to this as THE BASIC SERIES

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

Example:

Express $1/(1 + x^2)$ as a power series and find the interval of convergence.

Example (solution):

Replacing x by $-x^2$ in the basic series we have :

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

$$= \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \ldots$$

Example (solution):

Since this is a geometric series, it converges when $|-x^2| < 1$, that is, $x^2 < 1$, or $|x| < 1$.

Hence, the interval of convergence is $(-1, 1)$.

DIFFERENTIATION & INTEGRATION

Term-by-term theorem (part I):

If the power series $\Sigma\, c_n(x - a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n$$

is <u>differentiable</u> (and therefore continuous) on the interval $(a - R, a + R)$.

Term-by-term theorem (part II):

Also,

$$f'(x)=\sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

and

$$\int f(x)\,\mathrm{d}x = C+\sum_{n=0}^{\infty}\frac{c_n}{n+1}(x-a)^{n+1}$$

The radius of convergence of these power series is (again) R.

Example 1:

Express $1/(1 - x)^2$ as a power series by differentiating the basic series.

What is the radius of convergence?

Example 1 (solution):

Differentiating each side of the equation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$$

we get:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots = \sum_{n=1}^{\infty} nx^{n-1}$$

This series has radius R = 1 similar to the basic series

Example 2:

Find a power series representation for $ln(1 - x)$ and its radius of convergence.

Solution:

We notice that, except for a factor of –1, the derivative of this function is $1/(1 - x)$.

Example 2 (continued):

We integrate both sides of the basic series:

$$-\ln(1-x) = \int \frac{1}{1-x}\,dx$$

$$= \int (1 + x + x^2 + \ldots)\,dx$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots + C$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C = \sum_{n=1}^{\infty} \frac{x^n}{n} + C \qquad |x| < 1$$

Example 2 (continued):

To determine the value of C, we put $x = 0$ in this equation and obtain $-ln(1 - 0) = C$.

Thus, $C = 0$ and

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots = -\sum_{n=1}^{\infty} \frac{x^n}{n} \qquad |x| < 1$$

- The radius of convergence is the same as for the original series: $R = 1$.

Example 2 (continued):

Notice what happens if we put $x = ½$ in the result.

- Since ln ½ = -ln 2, we see that:

$$\ln 2 = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n2^n}$$

Taylor and Maclaurin Series

Brook Taylor (1685–1731)

Colin Maclaurin (1698–1746)

TAYLOR & MACLAURIN SERIES

Taylor series are useful for:

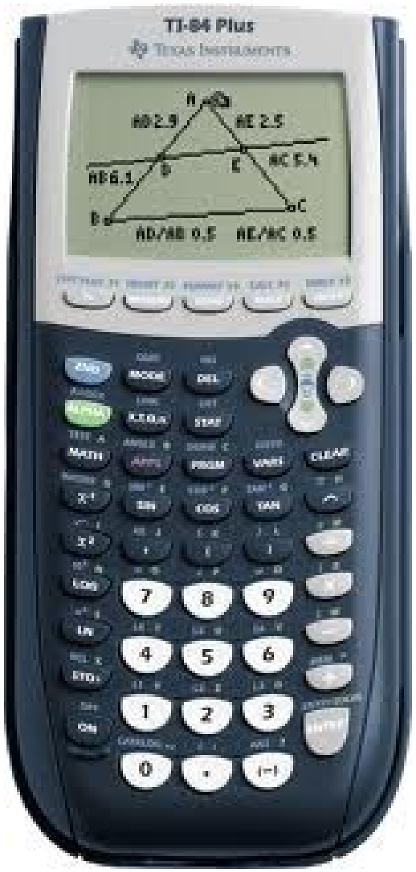
- Integrating functions without elementary antiderivatives.
- Solving differential equations.
- Approximating functions by polynomials.

CALCULATE sin(1)

Problem: calculate sin(1)

Solution: get your pocket calculator …

Question: your pocket calculator can only add and multiply finitely many numbers; so how does “he” do it?



CALCULATE sin(1)

$$f(x) = \sin x$$

<u>Taylor-polynomials:</u>

$P_0(x) = 0$ $\qquad \sin 1 \approx P_0(1) = 0$

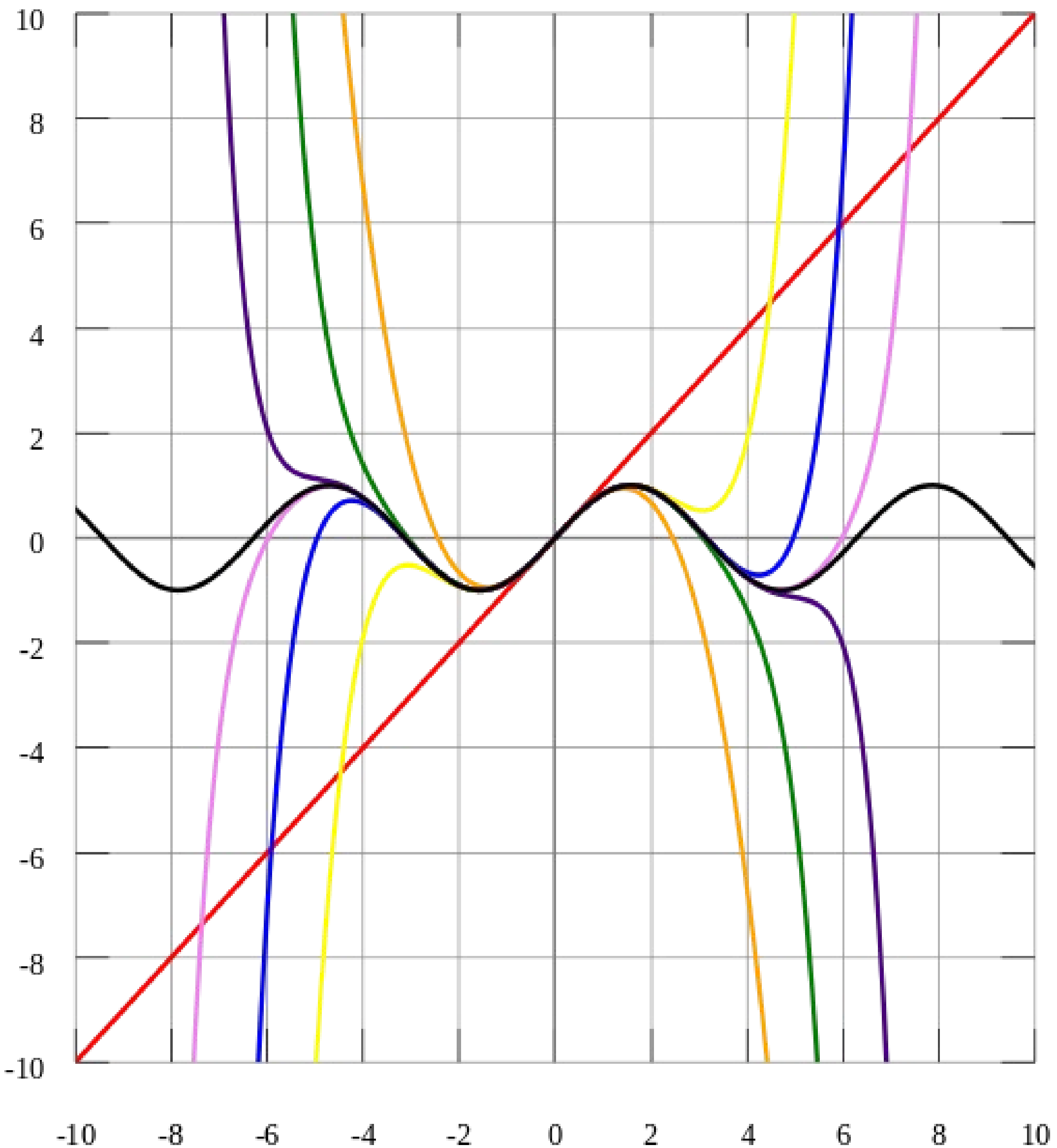
$P_1(x) = x$ $\qquad \sin 1 \approx P_1(1) = 1$

$P_3(x) = x - \frac{1}{6}x^3$ $\qquad \sin 1 \approx P_3(1) = 1 - \frac{1}{6} = \underline{0.8}33333...$

$P_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$ $\qquad \sin 1 \approx P_5(1) = 1 - \frac{1}{6} + \frac{1}{120} = \underline{0.841}67...$

$P_7(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7$ $\qquad \sin 1 \approx P_7(1) = (4\ terms) = \underline{0.84146}8...$

$$\sin 1 = 0.841471...$$



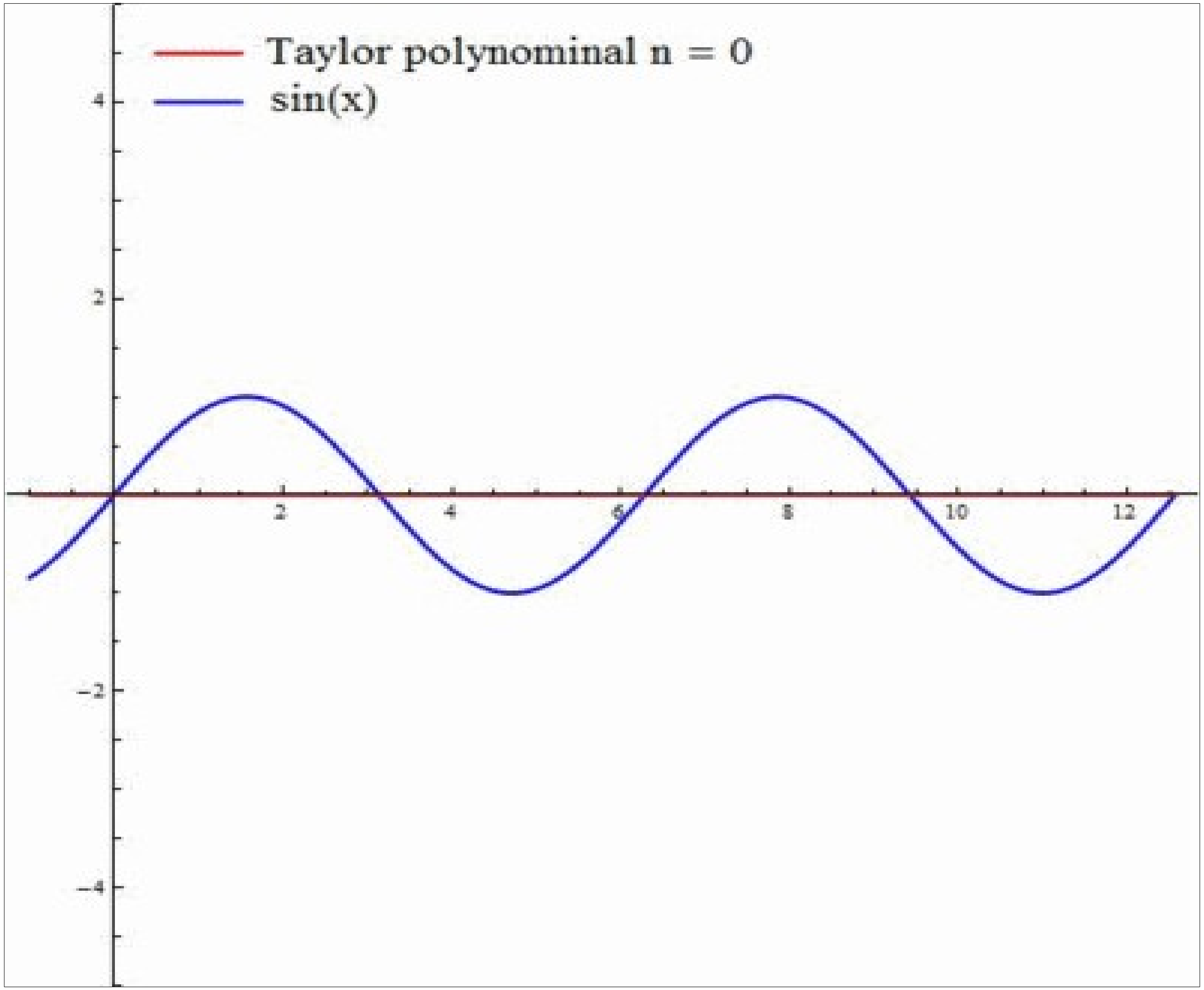
red: P_1

orange: P_3

yellow: P_5

green: P_7

TAYLOR & MACLAURIN SERIES



Theorem:

If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n \qquad |x - a| < R$$

then its coefficients are given by:

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Definition (Taylor series):

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$+\frac{f'''(a)}{3!}(x-a)^3+\cdots$$

This series is called the ***Taylor series of the function*** f ***at*** a (or about a or centered at a).

Definition (Maclaurin series):

For the special case $a = 0$, the Taylor series becomes:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$$

This is called the ***Maclaurin series***.

Example 1:

Find the Maclaurin series of the function $f(x) = e^x$.

Solution: $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = e^0 = 1$ for all n.

Hence, the Taylor series for f at 0
(that is, the Maclaurin series) is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Example 1 (continued):

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{for all } x$$

This series has an infinite convergence radius: $R = \infty$.

In particular, if we put $x = 1$, and obtain the following expression for the number e as a sum of an infinite series:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Example 2:

Find the Maclaurin series for sin *x*.

Solution: We arrange our computation in two columns:

$$
\begin{aligned}
f(x) &= \sin x & f(0) &= 0 \\
f'(x) &= \cos x & f'(0) &= 1 \\
f''(x) &= -\sin x & f''(0) &= 0 \\
f'''(x) &= -\cos x & f'''(0) &= -1 \\
f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0
\end{aligned}
$$

Example 2 (continued):

As the derivatives repeat in a cycle of four,
we can write the Maclaurin series for sin(x) as follows:

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Example 2 (continued):

It turns out that $R = \infty$. So:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \text{for all } x$$

Example 3: Find the Maclaurin series for cos x.

Solution: We could proceed directly.

However, differentiating the Maclaurin series for sin x:

$$\cos x = \frac{d}{dx}(\sin x)$$

$$= \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right)$$

$$= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Example 3 (continued):

Thus,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \text{for all } x$$

Example 4:

Find the Maclaurin series for the function
$f(x) = x \cos x$.

Solution:

Instead of computing derivatives and substituting it is easier to multiply the series for $\cos x$ by x:

$$x \cos x = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1) \frac{x^{2n+1}}{(2n)!}$$

Table 1:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

Table 1 (continued):

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\left(\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1\right)$$

Taylor polynomials

TAYLOR POLYNOMIALS

Definition:

The partial sum of the Taylor series up to order *n*:

$$P_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$+ \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called P_n the ***Taylor polynomial of order*** n ***around*** $x = a$.

Taylor polynomial of order 1

The 1st order Taylor polynomial of *f* around *a*, is given by:

$$P_1(x) = f(a) + f'(a)(x - a)$$

(This is exactly the linearization of *f* around $x = a$)

Note that: $P_1(a) = f(a)$

And also: $P_1'(x) = f'(a)$ so: $P_1'(a) = f'(a)$

So: the derivatives of P_1 and *f* agree up-to order 1.

Taylor polynomial of order 2

The 2nd order Taylor polynomial of f around a, is given by:

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

Note that: $P_2(a) = f(a)$

And: $P_2'(x) = f'(a) + f''(a)(x-a)$

So still : $P_2'(a) = f'(a)$

Finally: $P_2''(x) = f''(a)$ so: $P_2''(a) = f''(a)$

So: the derivatives of P_2 and f agree up-to order 2

Taylor polynomial of order n

The n[th] order Taylor polynomial of *f* around *a*, is given by:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The derivatives of P_n and *f* agree up-to order *n*.

The function P_n is the only polynomial of degree *n* for which the derivatives up-to order *n* agree with *f*.

TAYLOR POLYNOMIAL

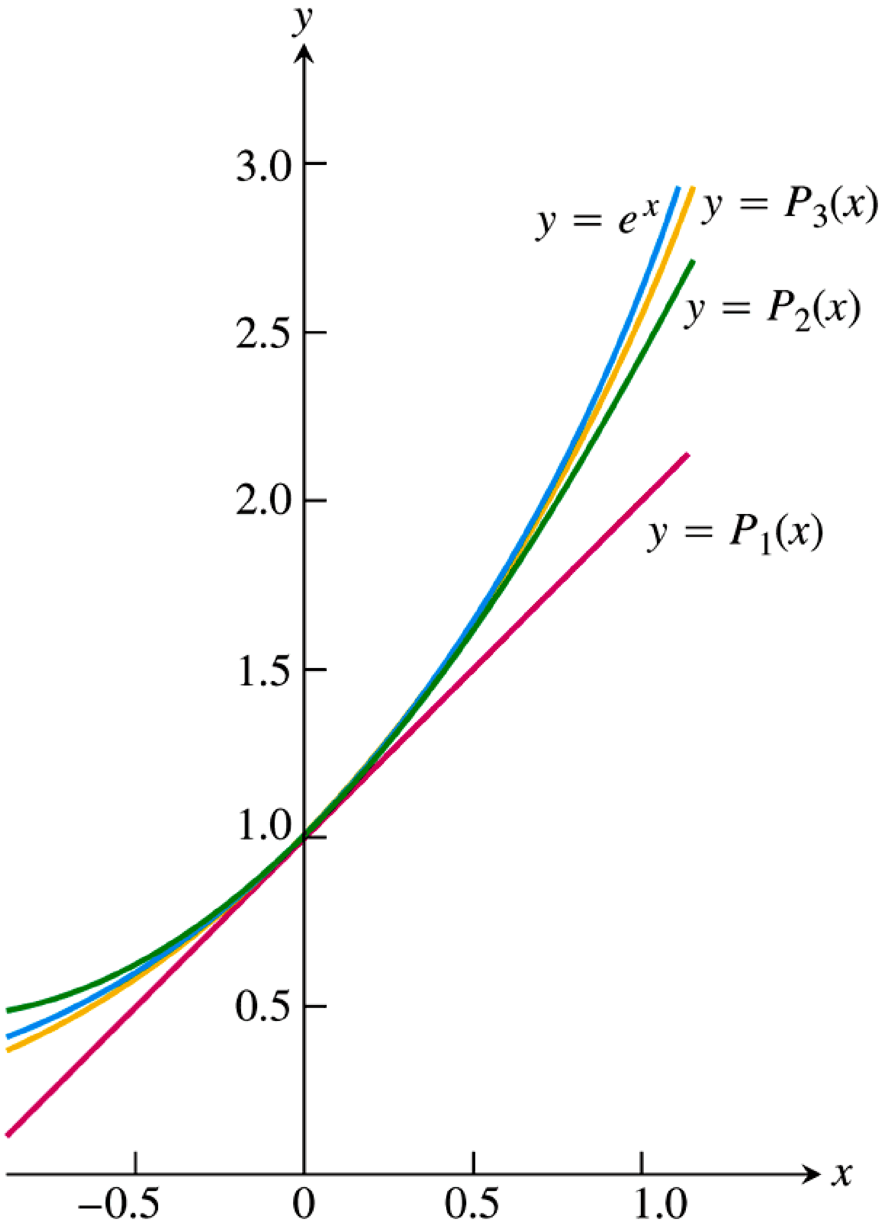


FIGURE 10.17 The graph of $f(x) = e^x$ and its Taylor polynomials

$$P_1(x) = 1 + x$$
$$P_2(x) = 1 + x + (x^2/2!)$$
$$P_3(x) = 1 + x + (x^2/2!) + (x^3/3!).$$

Notice the very close agreement near the center $x = 0$ (Example 2).

Basic Taylor polynomials (that you should know)

$$e^x : \quad 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!}$$

$$\cos x : \quad 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\sin x : \quad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\frac{1}{1-x} : \quad 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots + x^n$$

TAYLOR POLYNOMIAL

Using Basic Taylor polynomials

Example 1: $\sin x : \quad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

If x is close to 0 then also x^2 close is to 0, which leads to:

$$\sin(x^2) : x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots + (-1)^n \frac{x^{4n+2}}{(2n+1)!}$$

Summarizing Exercise

Determine the convergence interval of

$$\sum_{n=0}^{\infty}(x^2 - \frac{1}{2})^n$$

Compute the sum for all x in the convergence interval.

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