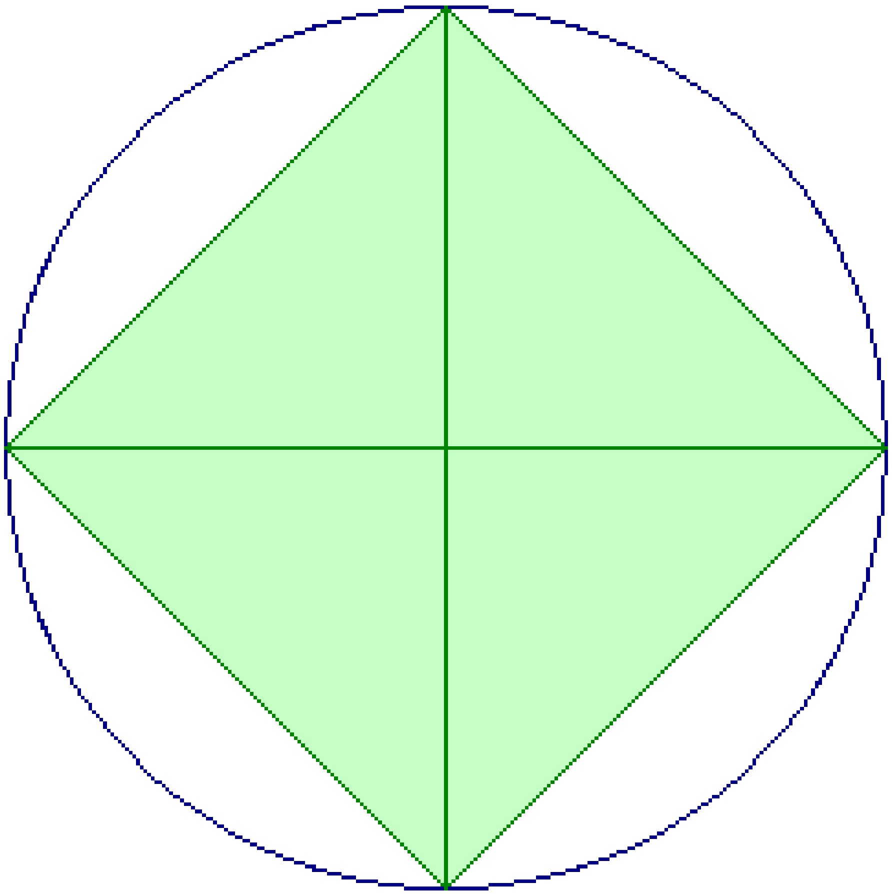
Mathematics B2: Newton

Area of circle as limit of triangle areas



Mathematics B2: Newton

-Contents-

- ✓ Integrals
- Calculation techniques for integrals
- Power and Taylor series

- First order ODEs
- Complex numbers
- Second order ODEs

Mathematics B2: Newton

- √ Substitution method
- Integration by parts
- Improper integrals

Isaac Newton

5.5

Indefinite Integrals and the substitution method

Example:

Find: $$\int \frac{\cos x}{\sin x}\, dx$$

- We first show how the substitution $\sin(x) = u$ works here.
- Then we prove that it is mathematically correct
- Finally we explain how we found this substitution.

Introduction Substitution Method $\int \frac{\cos x}{\sin x}\,dx$

If we substitute $\sin(x) = u$, what to do with $\cos(x)$ and dx??

We try to find out by "calculating" du.

We have:

$$u = \sin x \quad \text{so} \quad \frac{du}{dx} = \cos x \quad \text{and} \quad du = \cos x\, dx$$

This suggests to replace "cos x dx" by "du" !

sin(x) = u and cos x dx = du leads to

$$\int \frac{\cos x}{\sin x}\, dx = \int \frac{1}{\sin x} \cos x \, dx = \int \frac{1}{u}\, du = \ln|u| + C$$

$$= \ln|\sin x| + C$$

Check by differentiation:

$$\frac{d}{dx}(\ln|\sin x| + C) = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

Substitution Method

THEOREM 6—The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

$$[\,= F(u) + C = F(g(x)) + C\,]$$

Check by differentiation:

$$\left(F(g(x)) + C\right)' = F'(g(x)) \cdot g'(x) = f(g(x)).g'(x)$$

Substitution Method

Strategy: replace the expression you don't like by "u":

- u = e-power
- u = (square) root
- u = sin/cos/(tan)
- $\mathrm{d}u = u'(x)\mathrm{d}x$

Can lead to a simpler function in u…

Substitution Method

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Example:

Find: $\int \frac{e^x}{e^x + 1}\,dx$

Solution:

We want to get rid of e^x.

Hence $u = g(x) = e^x$; $du = e^x\,dx$; moreover $f(u) = \frac{1}{u+1}$

So: $\int \frac{e^x}{e^x+1}\,dx = \int \frac{1}{u+1}\,du = \ln|u+1| + C$

$$= \ln|e^x + 1| + C$$

Substitution Method

Another example:

sometimes a useful substitution is more difficult …

Find: $\int x^3 \sqrt{x^2+1}\, dx$

Solution: Just try the substitution: $u = \sqrt{x^2+1}$

Then: $\dfrac{du}{dx} = \dfrac{x}{\sqrt{x^2+1}}$ or: $dx = \dfrac{\sqrt{x^2+1}}{x}\, du$

Substituting gives:

$$\int x^3 \sqrt{x^2+1}\, dx = \int x^3 u \frac{\sqrt{x^2+1}}{x} du = \int x^2 u \sqrt{x^2+1}\, du$$

$$= \int x^2 u \cdot u\, du = \int x^2 u^2\, du$$

Solution (continued):

In $\int x^2u^2\,du$ **we have to express *x* in *u* !!**

Now: $u=\sqrt{x^2+1}$ so $u^2=x^2+1$ so $x^2=u^2-1$

Hence: $\int x^3\sqrt{x^2+1}\,dx=\int x^2u^2\,du=\int(u^2-1)u^2\,du=\int\left(u^4-u^2\right)du$

$$=\frac{1}{5}u^5-\frac{1}{3}u^3+C=\frac{1}{5}\left(\sqrt{x^2+1}\right)^5-\frac{1}{3}\left(\sqrt{x^2+1}\right)^3+C$$

You can check the result by differentiation.

5.6

Substitution Method
(Definite integrals)

Substitution Method

The substitution method for definite integrals is:

$$\int_a^b f\big(g(x)\big)\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Proof:

According to the substitution method for indefinite integrals:

$$\int_a^b f\big(g(x)\big)\cdot g'(x)\,dx = \Big[F\big(g(x)\big)\Big]_{x=a}^{x=b} = F\big(g(b)\big) - F\big(g(a)\big)$$

Also:

$$\int_{g(a)}^{g(b)} f(u)\,du = \Big[F(u)\Big]_{u=g(a)}^{u=g(b)} = F\big(g(b)\big) - F\big(g(a)\big)$$

Substitution Method

$$\int_a^b f\left(g(x)\right) \cdot g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

<u>So there are three changes</u>:

1. The integrand changes (*via* $u = g(x)$)
2. The differential dx changes (dx = d$u/g'(x)$)
3. The limits of integration change

Substitution Method

$$\int_a^b f(g(x))\cdot g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Example: calculate $\int_0^{\frac{\pi}{4}} \tan x dx$

Solution: Because $\tan x = \frac{\sin x}{\cos x}$ we substitute $u = \cos(x)$

So du = -sin(x) dx, and hence sin(x) dx = -du

So:

$$\int_0^{\frac{\pi}{4}} \tan x dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} dx = \int_{\cos 0}^{\cos(\frac{\pi}{4})} -\frac{du}{u} = \int_1^{\frac{\sqrt{2}}{2}} -\frac{1}{u} du =$$

$$= \left[-\ln|u|\right]_{u=1}^{u=\frac{\sqrt{2}}{2}} = -\ln\frac{\sqrt{2}}{2} + \ln 1 = -\ln\frac{\sqrt{2}}{2}$$

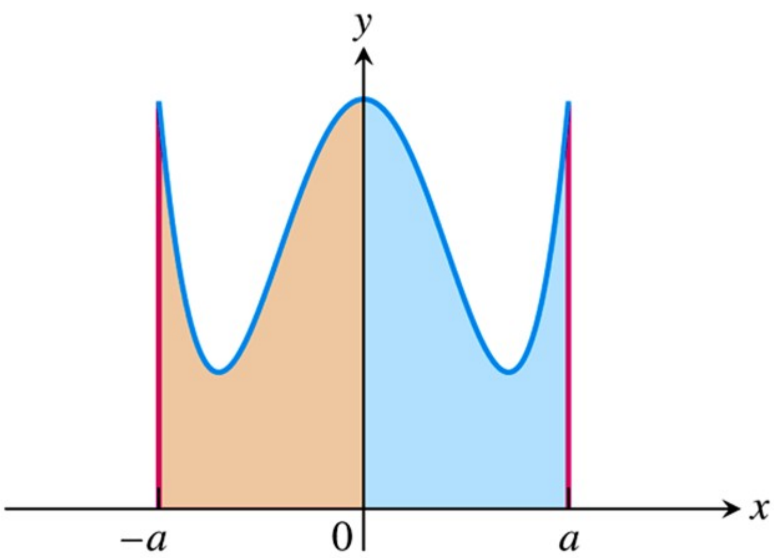
Even and odd functions

Odd and even functions

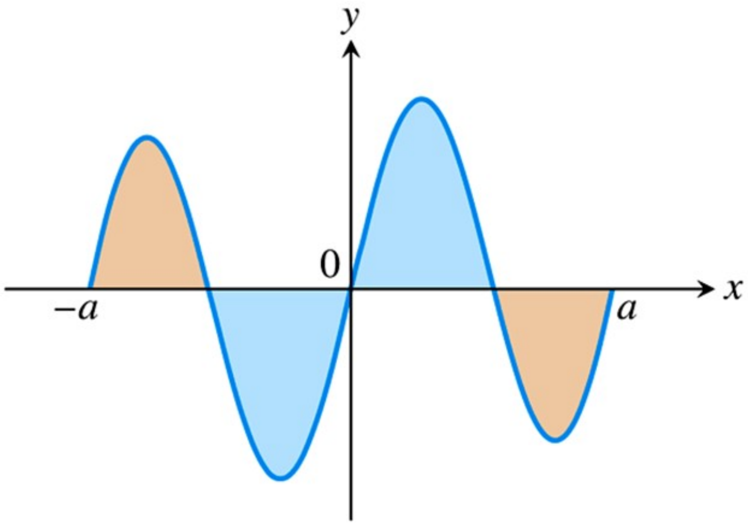
THEOREM 8 Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.



f is even



f is odd

5.6

Area between curves

Area between curves

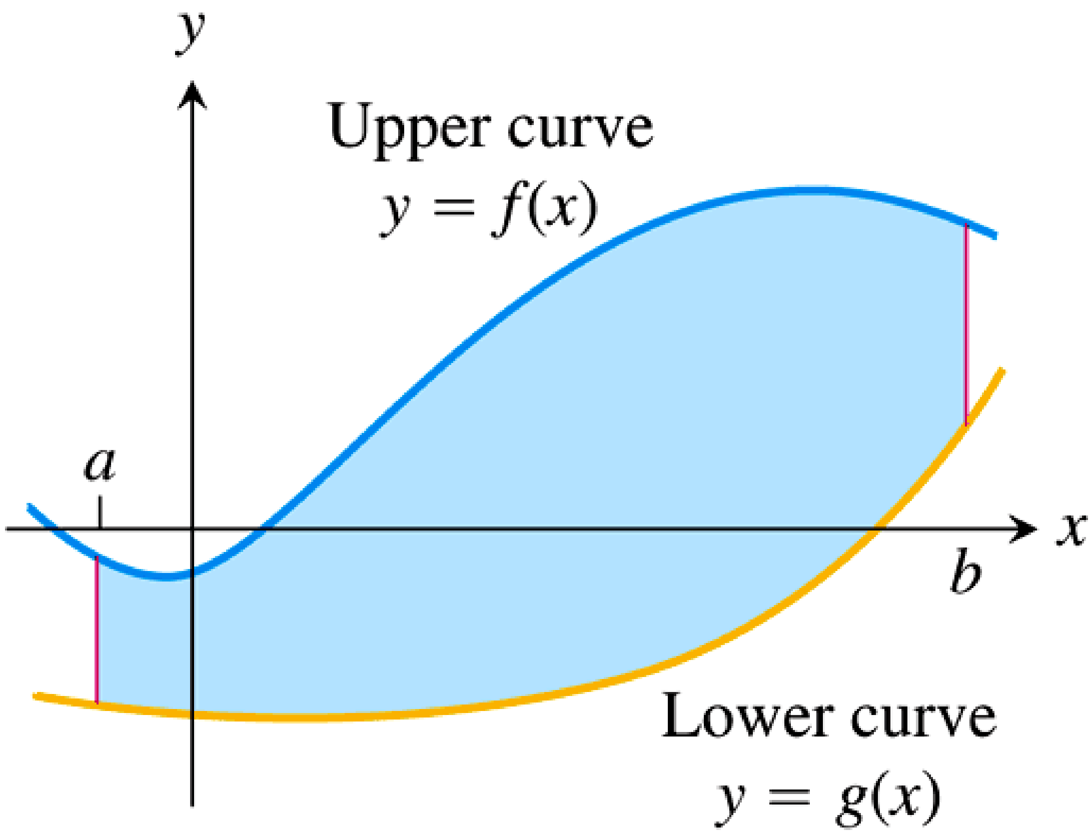


FIGURE 5.25 The region between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

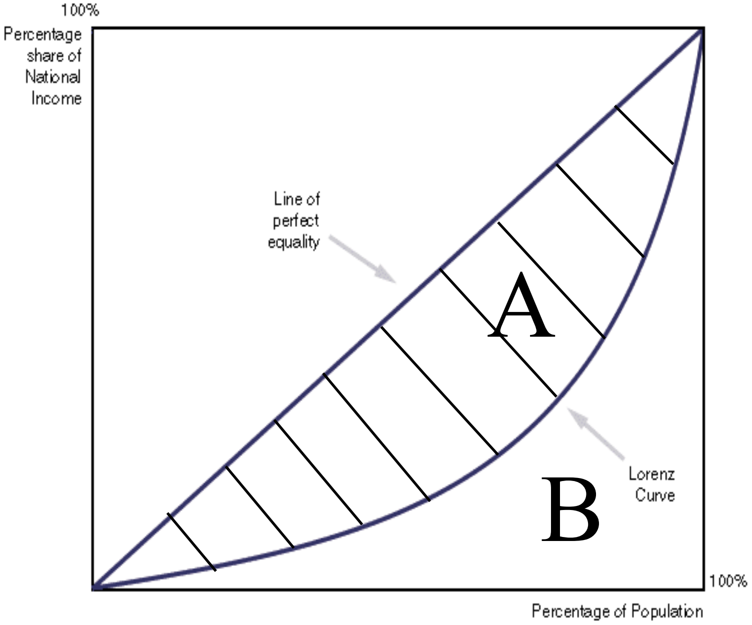
Area between curves

DEFINITION If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

Example: Gini index

$$G = 2\int_0^1 (x - L(x))dx$$



Mathematics B2: Newton

- [x] Substitution method
- [x] Integration by parts
- [ ] Improper integrals

Isaac Newton

8.1

Integration by Parts

Integration by parts

By definition of integral and derivative

$$f(x)g(x) = \int \frac{d}{dx}\Big(f(x)g(x)\Big)dx$$

… and by the product rule

$$= \int \Big(f'(x)g(x) + f(x)g'(x)\Big)dx$$

Integration by parts

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \qquad (1)$$

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du \qquad (2)$$

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx \qquad (3)$$

Example:

$$\int x^2 e^x \, dx .$$

Solution:

We try: $u = x^2$ and $dv = e^x dx$ (then du is simpler than u)

So: $du = 2x\, dx$ en $v = e^x$

Hence: $\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx$

$$\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx$$

Solution (continued):

Now take: $u = 2x$ and $dv = e^x \, dx$
(note that du again gets simpler)

Then: $du = 2 \, dx$ en $v = e^x$

So: $x^2 e^x - \int 2x e^x \, dx = x^2 e^x - \left(2x e^x - \int 2 e^x \, dx\right)$

$$= x^2 e^x - 2x e^x + 2e^x + C$$

Finally: $\int x^2 e^x \, dx = x^2 e^x - 2x e^x + 2e^x + C$

Integration by parts

Example: evaluate $\int x \sin x dx$

Solution:

Take $u = x$ and $dv = \sin(x)dx$
Then $du = dx$ and $v = -\cos(x)$

We conclude:

$$\int x \sin x dx = -x \cos x - \int -\cos x dx$$
$$= -x \cos x + \sin x + C$$

Find: $\int \ln x dx$

Solution:

Use partial integration with $u = \ln(x)$ and $dv = 1dx$.
Then $du = 1/x\, dx$ and $v = x$.

We conclude:

$$\int \ln x dx = x \ln x - \int x \cdot \frac{1}{x} dx$$
$$= x \ln x - x + C$$

Example: Find $\int x^2 \ln x \, dx$

Solution:

$u = \ln(x)$ and $dv = x^2 dx$

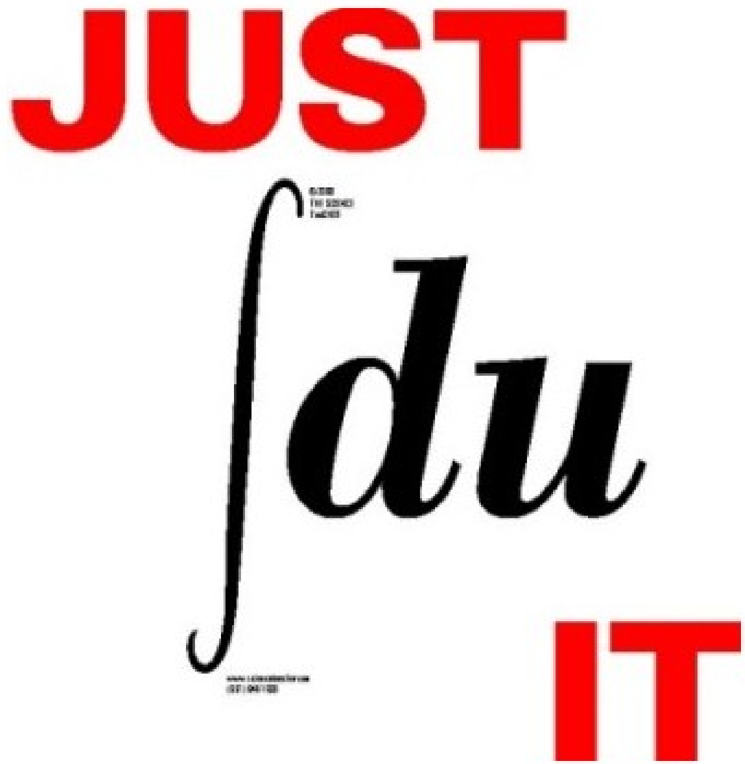
Then: $du = 1/x \, dx$ en $v = 1/3 \, x^3$

And:

$$\int x^2 \ln x \, dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \cdot \frac{1}{x} dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^2 \, dx$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$$

Break



8.7

Improper Integrals

Improper integrals

In this section, we extend the concept of a definite integral to the cases where:

- the interval is infinite , thus $[a,\infty)$ or $(-\infty,b]$.
- f has a vertical asymptote in $[a, b]$.

In either case, the integral is called an ***improper integral***.

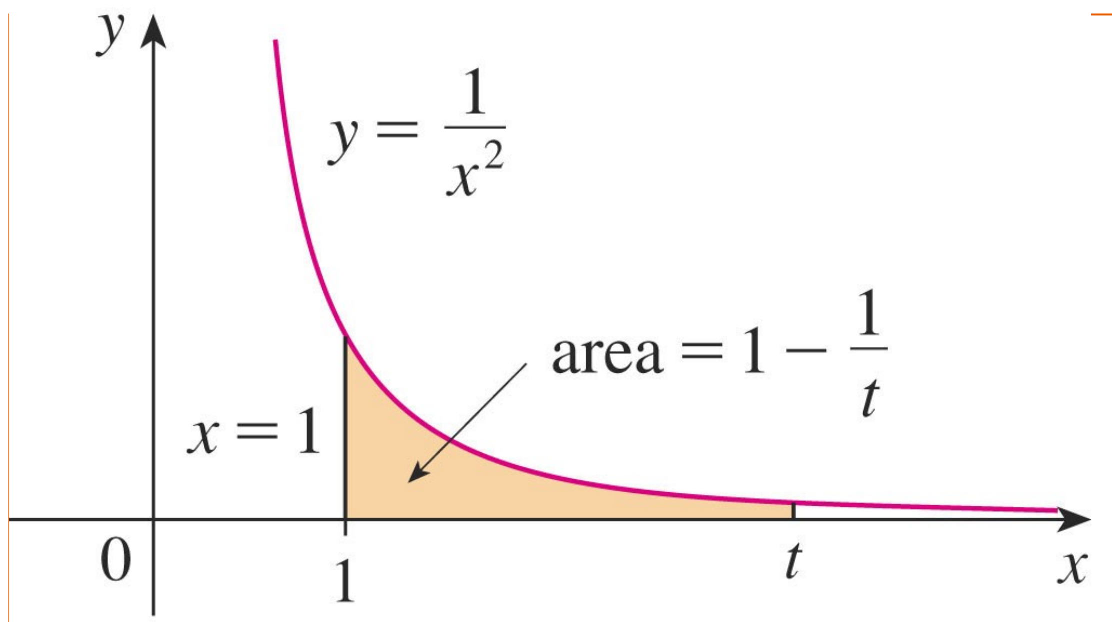
Consider the infinite region S that lies:

- Under the curve $y = 1/x^2$
- Above the x-axis
- To the right of the line $x = 1$

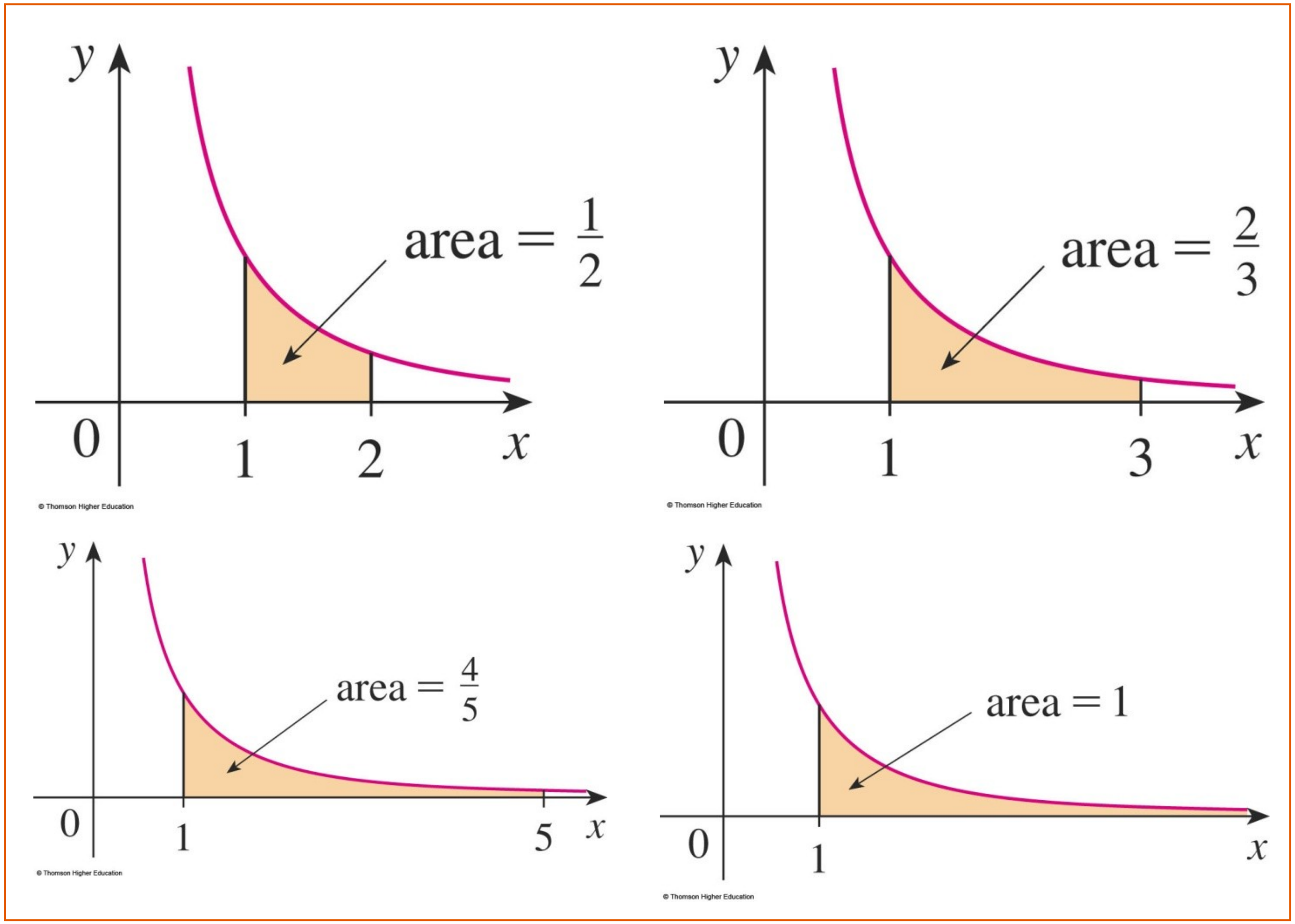
The area of the part of S that lies to the left of the line $x = t$ (shaded) is:

$$A(t) = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg]_1^t = 1 - \frac{1}{t}$$

- Notice that $A(t) < 1$ no matter how large t is chosen.



The area of the shaded region approaches 1 as $t \to \infty$.



We observe that:

$$\lim_{t \to \infty} A(t) = \lim_{t \to \infty} \left(1 - \frac{1}{t} \right) = 1$$

So, we say that the area of the infinite region S is equal to 1 and we write:

$$\int_1^{\infty} \frac{1}{x^2}\, dx = \lim_{t \to \infty} \int_1^{t} \frac{1}{x^2}\, dx = 1$$

Definition improper integral

If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t \to \infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^{b} f(x)dx = \lim_{t \to -\infty} \int_t^b f(x)dx$$

provided this limit exists (as a finite number).

Convergent and divergent

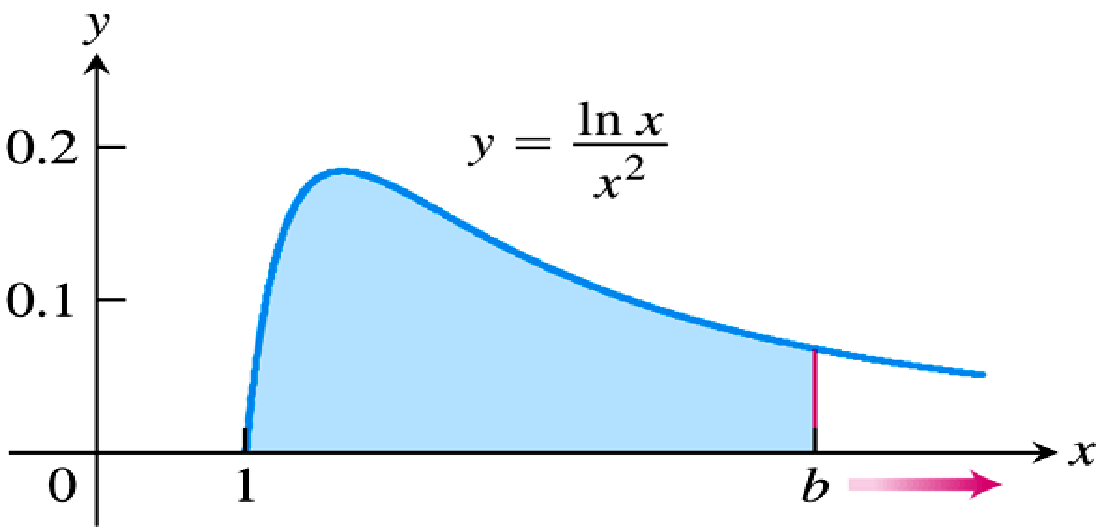


FIGURE 8.14 The area under this curve is an improper integral (Example 1).

If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

Convergent and divergent

The improper integrals $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{b} f(x)\,dx$ are called:

- *Convergent* if the corresponding limit exists.
- *Divergent* if the limit does <u>not</u> exist.

Example:

Determine whether the integral

$$\int_1^{\infty} (1/x)\,dx$$

is convergent or divergent.

Example:

According to the definition,
we have:

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big]_1^t$$

$$= \lim_{t\to\infty} (\ln t - \ln 1)$$

$$= \lim_{t\to\infty} \ln t = \infty$$

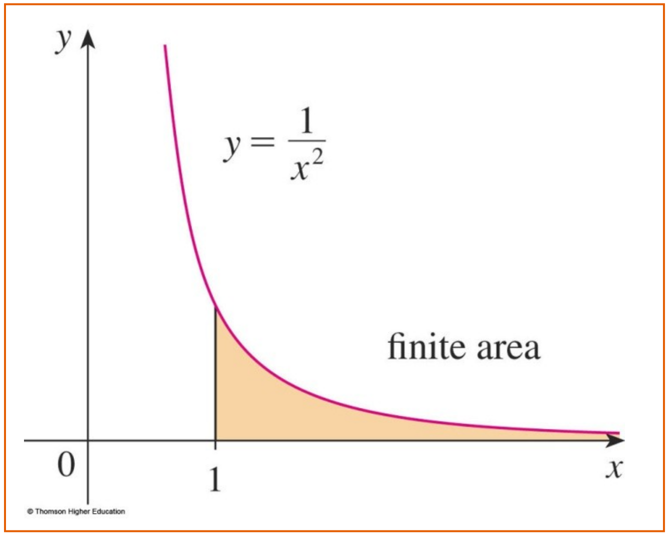
- The limit does not exist as a finite number.
- So, the integral is divergent.

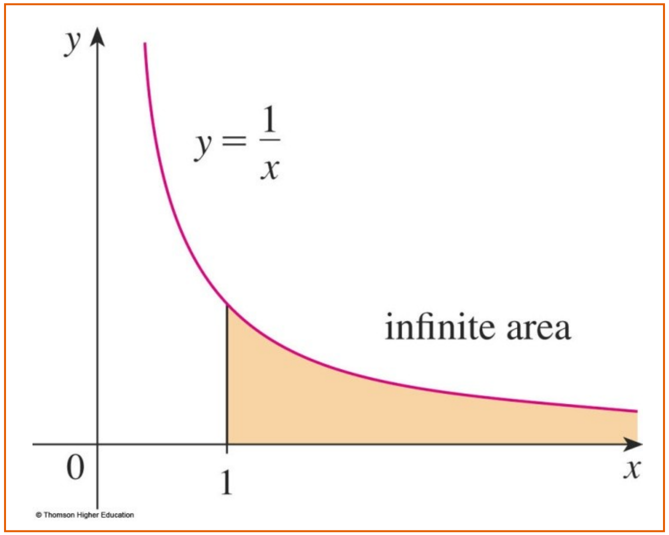
Convergent and divergent

Let's compare this result to
the example at the beginning of the section:

$$\int_1^\infty \frac{1}{x^2}\,dx \text{ converges} \qquad \int_1^\infty \frac{1}{x}\,dx \text{ diverges}$$

- Geometrically, this means the following.

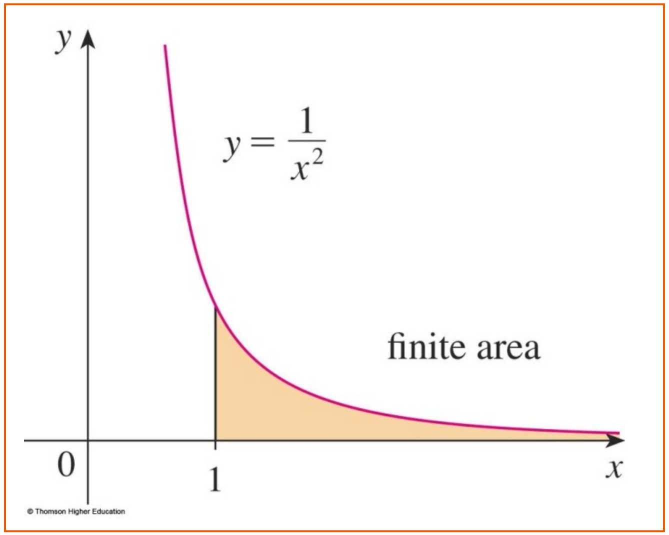


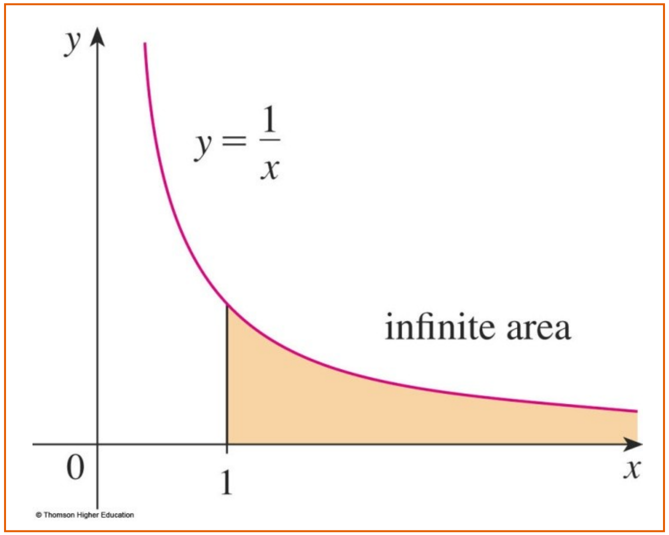


Convergent and divergent

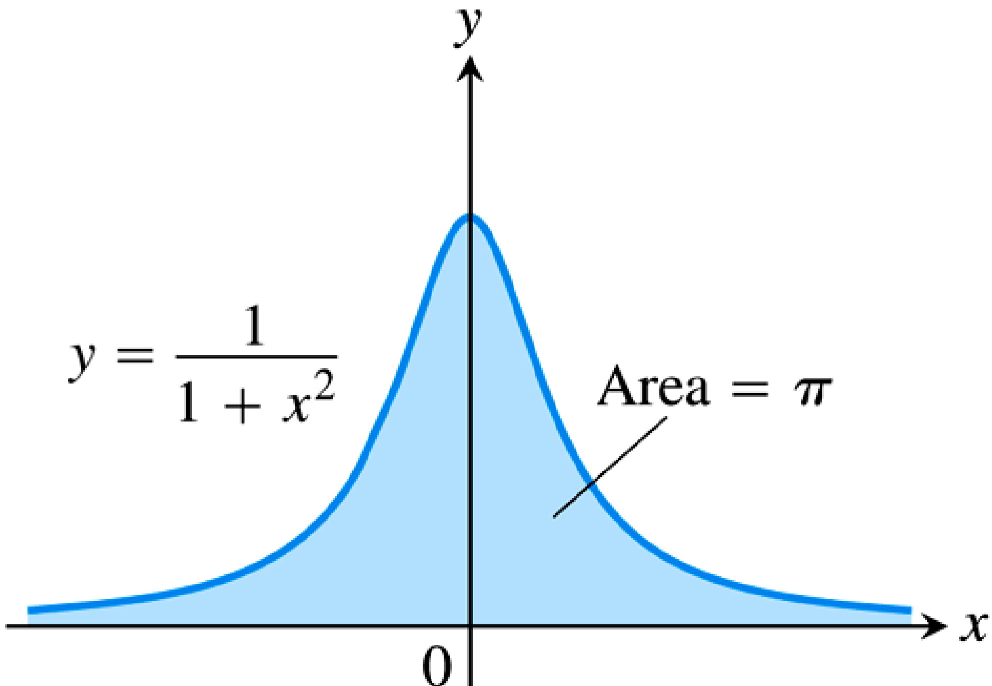
Note that both $1/x^2$ and $1/x$ approach 0 as $x \to \infty$, but $1/x^2$ approaches 0 faster than $1/x$.

- The function $1/x$ does not decrease fast enough for its integral to have a finite value.





Splitting integrals



If $f(x)$ is continuous on $(-\infty, \infty)$, then

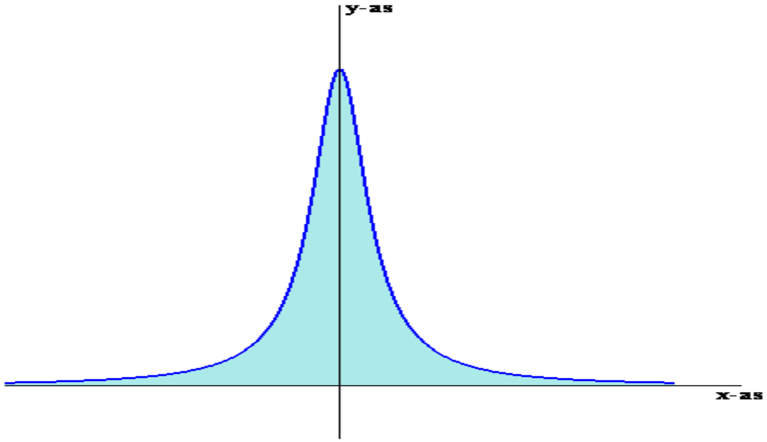
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx,$$

independent of the choice of c

Splitting integrals

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$$

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$$

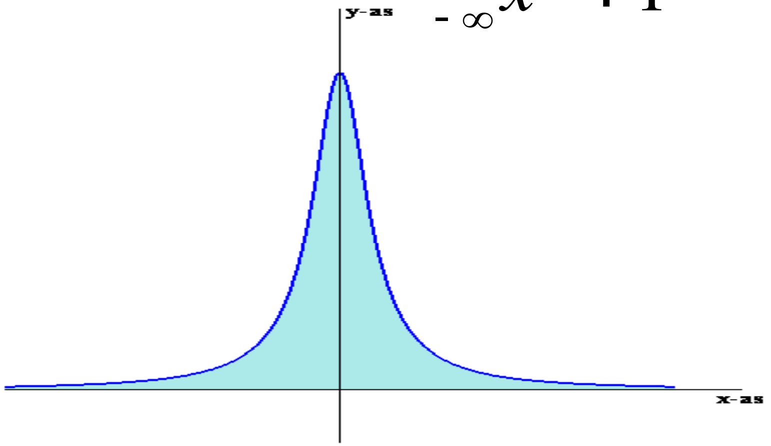


Solution: (Split)

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \int_{-\infty}^{0} \frac{1}{x^2+1}\,dx + \int_{0}^{\infty} \frac{1}{x^2+1}\,dx$$

Splitting integrals

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$$



$$\int_{-\infty}^{0} \frac{1}{x^2+1}\,dx = \lim_{t \to -\infty} \int_{t}^{0} \frac{1}{x^2+1}\,dx$$

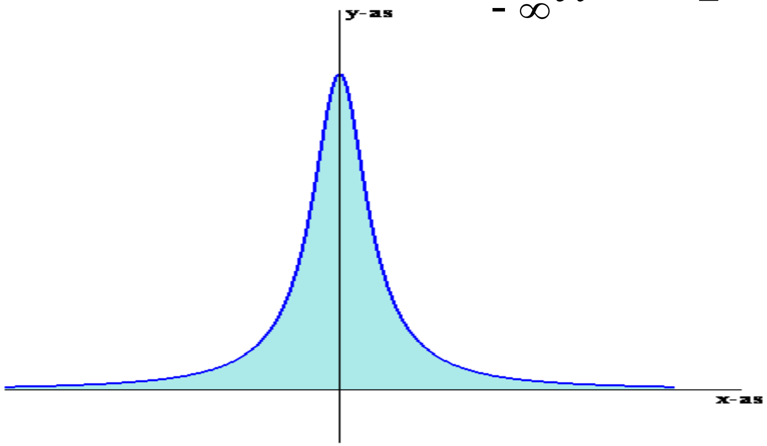
$$= \lim_{t \to -\infty} \left[\tan^{-1} x\right]_{x=t}^{x=0}$$

$$= \lim_{t \to -\infty} \left(\tan^{-1} 0 - \tan^{-1} t\right) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$\int_{0}^{t} \frac{1}{x^2+1}\,dx = \lim_{t \to \infty} \int_{0}^{t} \frac{1}{x^2+1}\,dx = \lim_{t \to \infty} \left[\tan^{-1} x\right]_{x=0}^{x=t}$$

$$= \lim_{t \to \infty} \left(\tan^{-1} t - \tan^{-1} 0\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Conclusion:



$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \int_{-\infty}^{0} \frac{1}{x^2+1}\,dx + \int_{0}^{\infty} \frac{1}{x^2+1}\,dx$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

DEFINITION Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx,$$

where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

Functions with 'bad' behavior

Whenever you see the symbol $\int_a^b f(x)\,dx$, you must decide, by looking at the function f on $[a, b]$, whether it is either:

- An ordinary definite integral
- An improper integral in which the integrand has, e.g., a **singularity**

Example: Evaluate

$$\int_0^1 \ln x \, dx$$

- We know that the function $f(x) = \ln x$ has a vertical asymptote at 0 since $\lim_{x\to 0^+} \ln x = -\infty$.

- Thus, the given integral is improper, and we have:

$$\int_0^1 \ln x \, dx = \lim_{t\to 0^+} \int_t^1 \ln x \, dx$$

Now, we integrate by parts with $u = \ln x$, $dv = dx$, $du = dx/x$, and $v = x$:

$$\int_t^1 \ln x \, dx = x \ln x \Big]_t^1 - \int_t^1 dx$$

$$= 1 \ln 1 - t \ln t - (1 - t)$$

$$= -t \ln t - 1 + t$$

Functions with ‘bad’ behavior

To find the limit of the first term, we use l’Hospital’s Rule:

$$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{1/t}$$

$$= \lim_{t\to 0^+} \frac{1/t}{-1/t^2}$$

$$= \lim_{t\to 0^+} (-t)$$

$$= 0$$

Therefore,

$$\int_0^1 \ln x \, dx = \lim_{t \to 0^+} (-t \ln t - 1 + t)$$

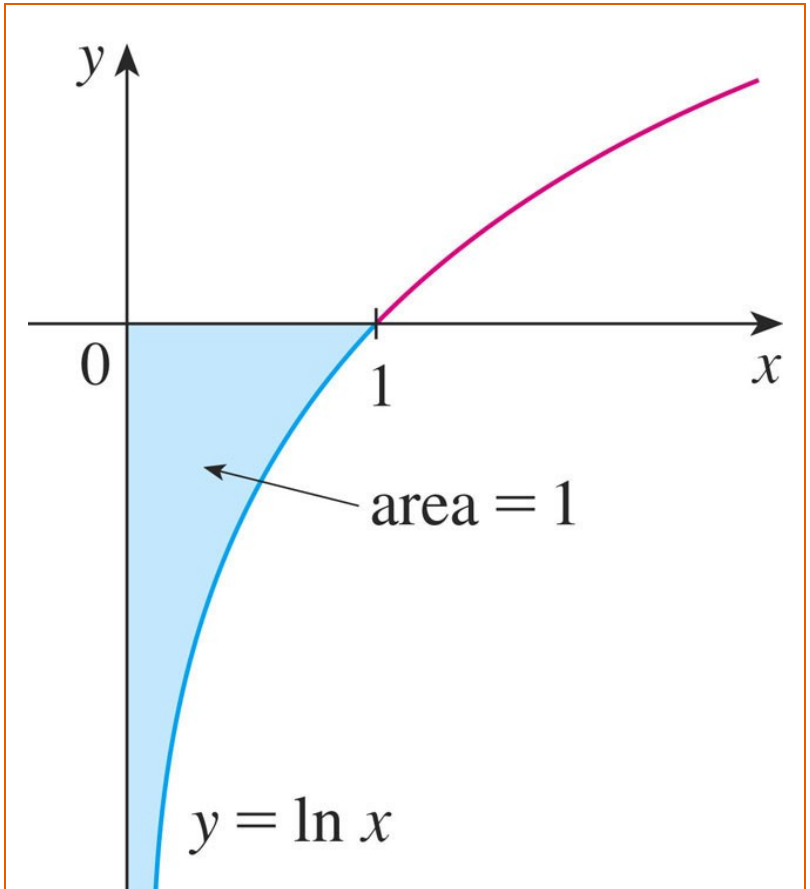
$$= -0 - 1 + 0$$

$$= -1$$

Functions with 'bad' behavior

The geometric interpretation of the result is shown.

- The area of the shaded region above $y = \ln x$ and below the x-axis is 1.



Functions with ‘bad’ behavior

DEFINITION Type II Improper Integrals

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II.**

1. If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

8.7

Terminology

Terminology

Definition for improper integrals of Type 1 or Type 2 :

If the limit(s) that define(s) the integral exist(s) (and are finite), then the integral is called ***convergent***.

If (one of) the limit(s) that define(s) the integral does not exist (or is infinite), then the integral is called ***divergent***.

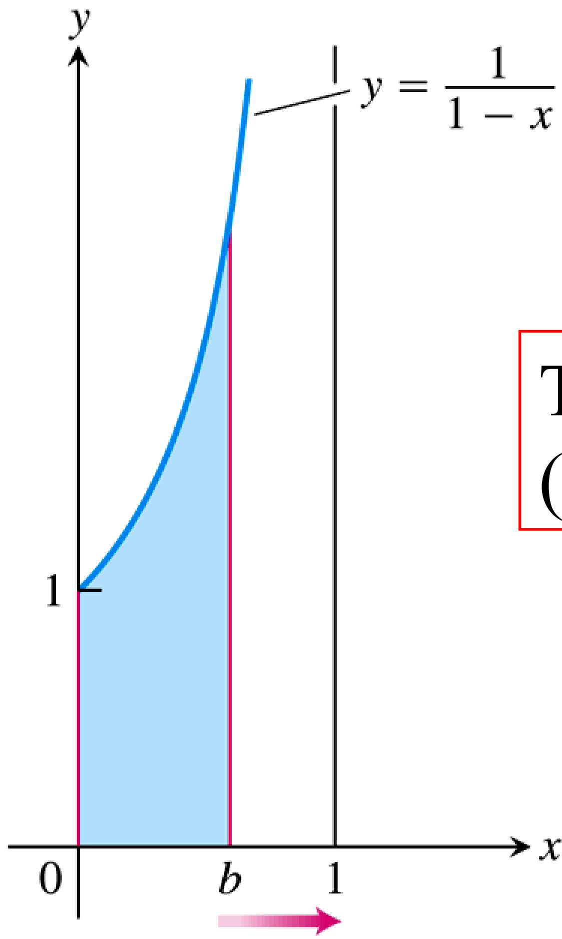
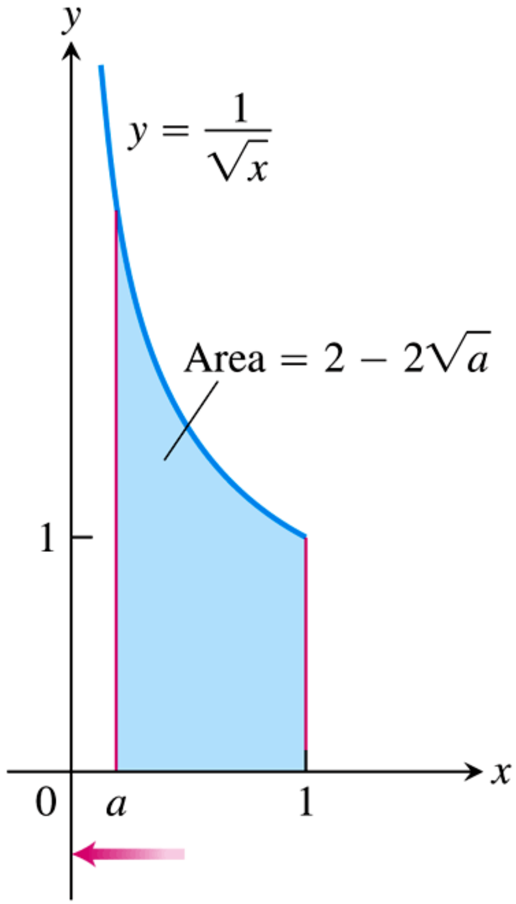


FIGURE 8.17 The area beneath the curve and above the x-axis for [0, 1) is not a real number

$$\to -\ln(1-x)$$

The integral is ***divergent***.
(the area is infinite)



$$\to 2\sqrt{x}$$

The integral is convergent (the area is finite)

Normal distributions

Estimating convergence/divergence of integrals

Show that $\int_0^{\infty} e^{-x^2}\,dx$ is convergent.

- We can't evaluate the integral directly.
- The antiderivative of e^{-x^2} is not an elementary function.

Estimating convergence/divergence of integrals

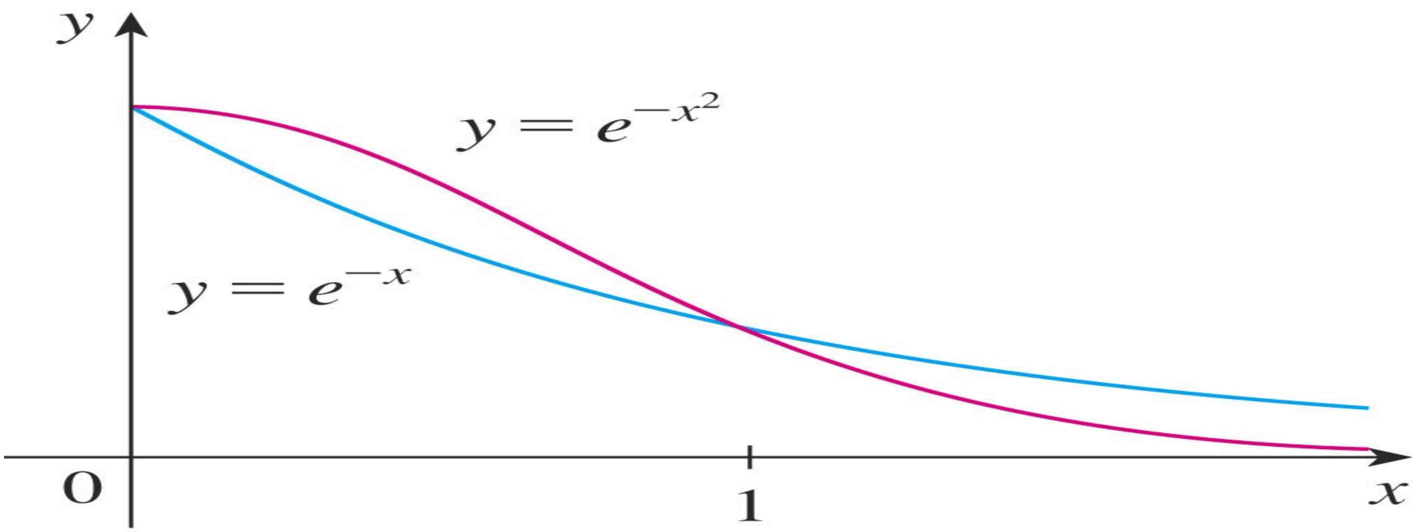
We write: $$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$$

- We observe that the first integral on the right-hand side is just an ordinary definite integral.

Estimating convergence/divergence of integrals

- In the second integral, we use the fact that, for $x \geq 1$, we have $x^2 \geq x$.
- So, $-x^2 \leq -x$ and, therefore, $e^{-x^2} \leq e^{-x}$.
- This also holds for the integrals

$$0 <= \int_1^{\infty} e^{-x^2}\,dx <= \int_1^{\infty} e^{-x}\,dx$$



The integral of e^{-x} is easy to evaluate:

$$\int_1^{\infty} e^{-x}\,dx = \lim_{t\to\infty}\int_1^{t} e^{-x}\,dx$$

$$= \lim_{t\to\infty}(e^{-1} - e^{-t})$$

$$= e^{-1}$$

It follows that $\int_0^{\infty} e^{-x^2}\,dx$ is convergent.

Estimating convergence/divergence of integrals

We showed that $\int_0^{\infty} e^{-x^2} dx$

is convergent without computing its value.

- In probability theory, it is important to know the exact value of this improper integral.

- It can be shown that the exact value is $\sqrt{\pi}/2.$
  - this is not a subject of this course

Normal distributions

Many important random phenomena are modeled by a normal distribution.

Examples are:

- Test scores on aptitude tests
- Heights and weights of individuals from a homogeneous population
- Annual rainfall in a given location

Normal distributions

This means that the probability density function of the random variable *X* is a member of the family of functions

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

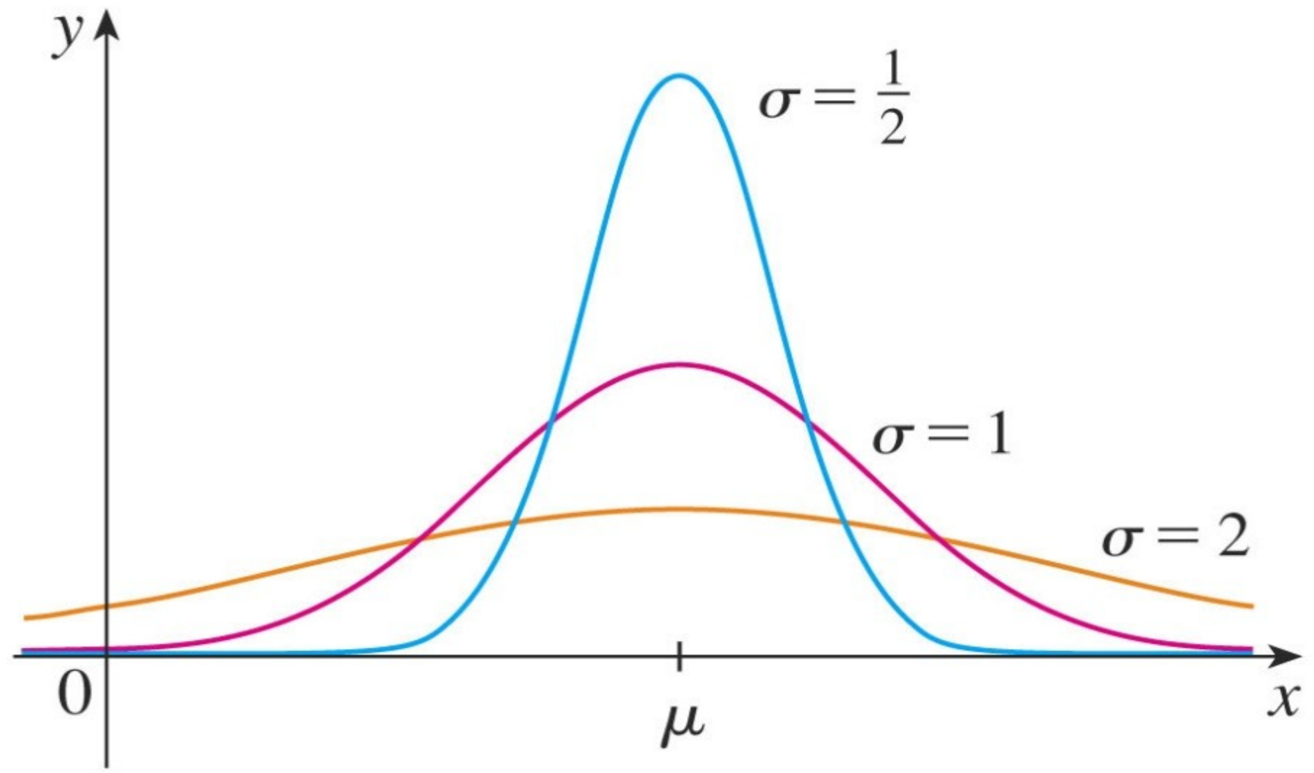
The factor $1/(\sigma\sqrt{2\pi})$ is needed to make *f* a probability density function:

$$\int_{\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} dx = 1$$

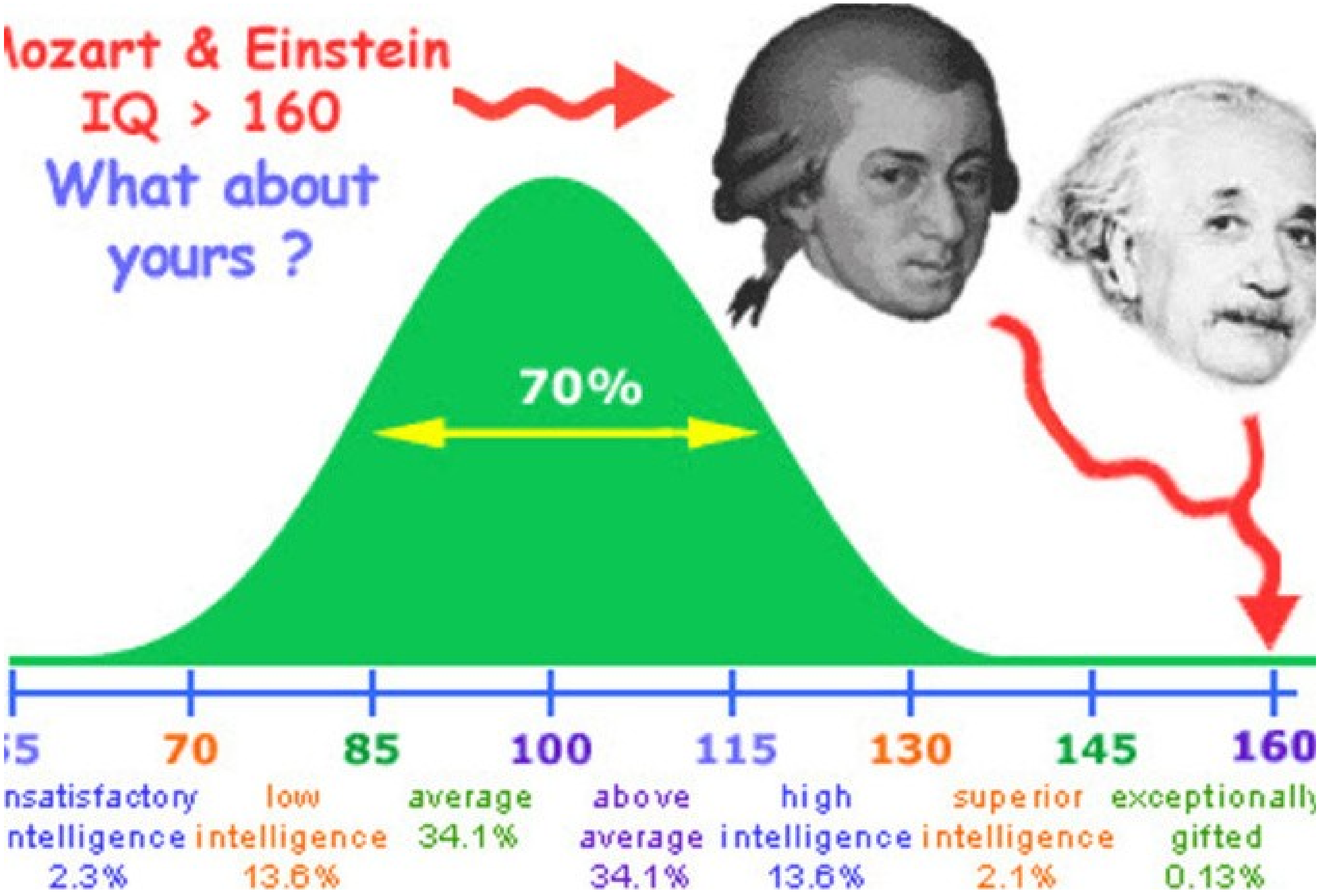
Normal distributions

From graphs of members of the family, we see that:

- For small values of σ, the values of X are clustered about the mean μ.
- For larger values of σ, the values of X are more spread out.



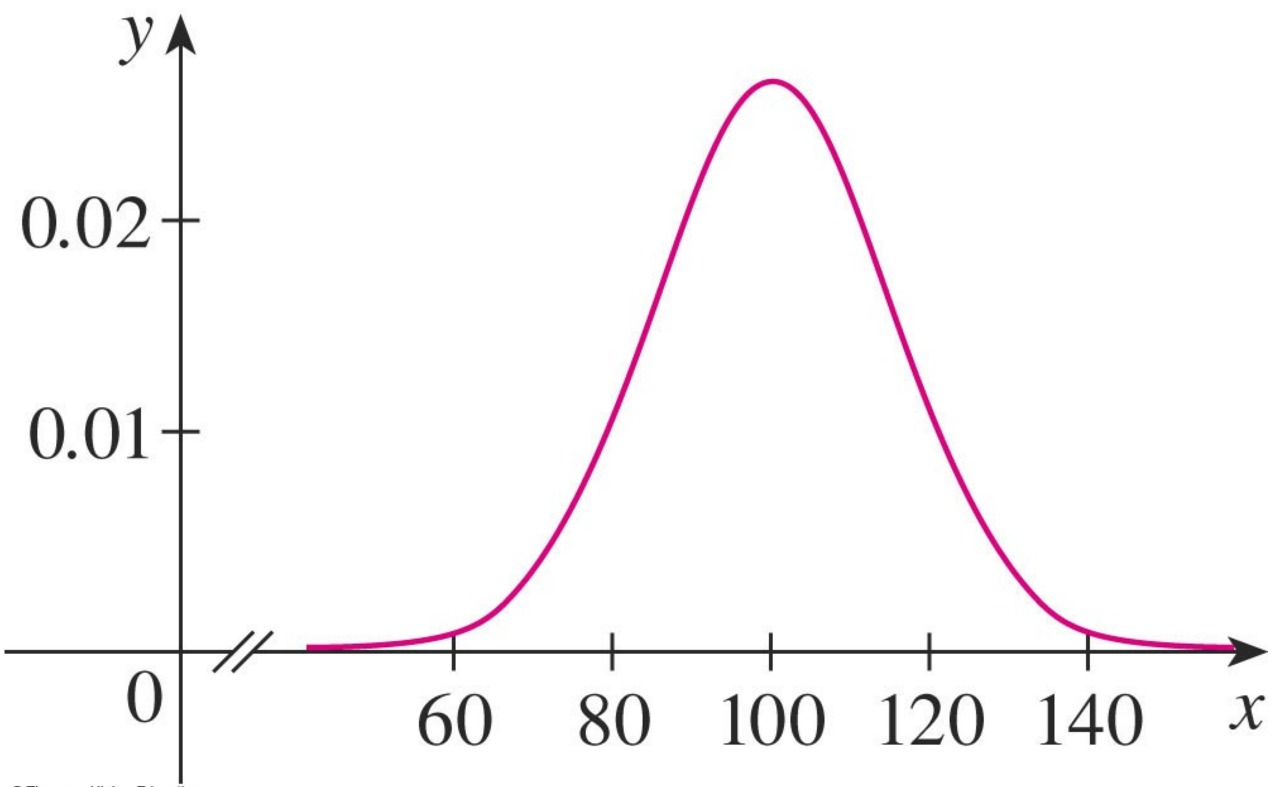
Intelligence Quotient



Intelligence Quotient

Intelligence Quotient (IQ) scores are distributed normally with mean 100 and standard deviation 15.

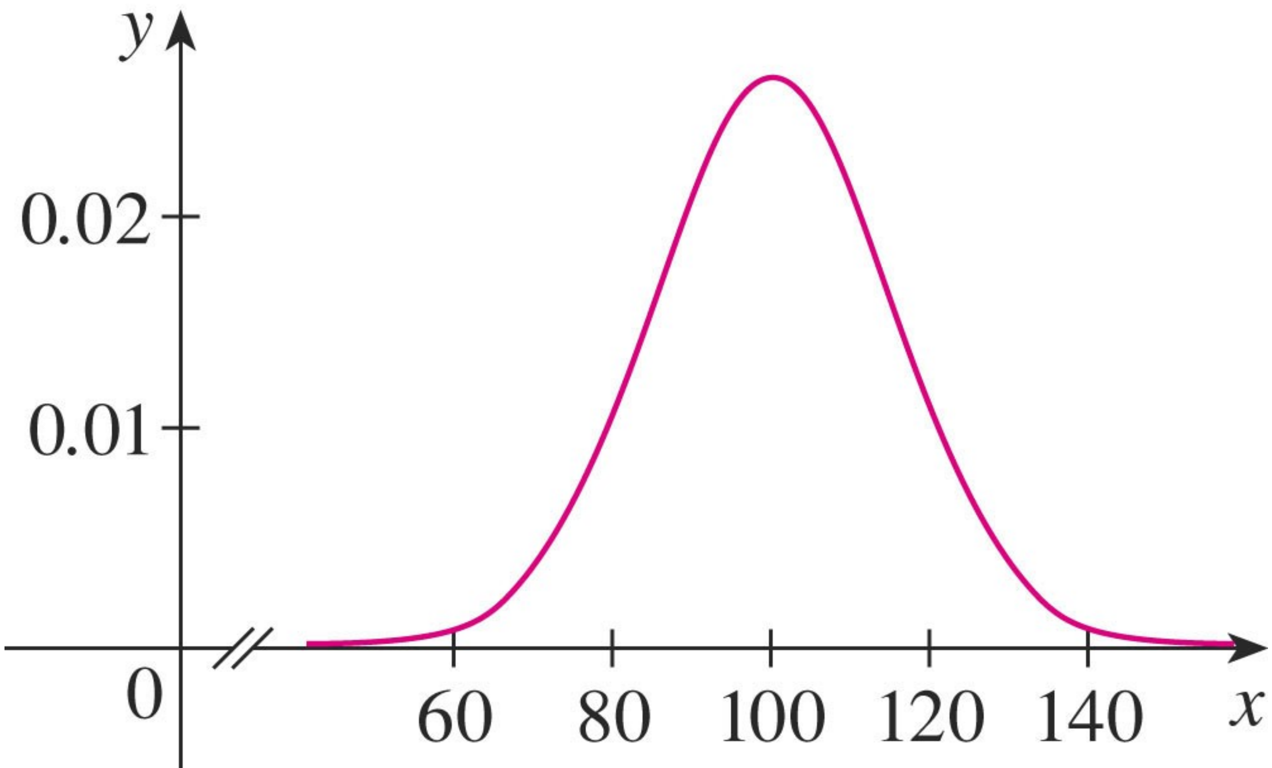
- The figure shows the corresponding probability density function.



Intelligence Quotient

a. What percentage of the population has an IQ score between 85 and 115?

b. What percentage has an IQ above 140?



Intelligence Quotient

As IQ scores are normally distributed, we use the probability density function given by Equation 3 with $\mu = 100$ and $\sigma = 15$:

$$P(85 \leq X \leq 115)$$

$$= \int_{85}^{115} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/450)} dx$$

Intelligence Quotient

Recall that the function $y = e^{-x^2}$ doesn't have an elementary antiderivative.

- So, we can't evaluate the integral exactly.

However, we can use the numerical integration capability of a calculator or computer (or the Midpoint Rule) to estimate the integral.

Doing so, we find that:

$$P(85 \leq X \leq 115) \approx 0.68$$

About 68% of the population has an IQ between 85 and 115—that is, within one standard deviation of the mean.

Intelligence Quotient

The probability that the IQ score of a person chosen at random is more than 140 is:

$$P(X > 140) = \int_{140}^{\infty} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/450} dx$$

To avoid the improper integral, we could approximate it by the integral from 140 to 200.

- It's quite safe to say that people with an IQ over 200 are extremely rare.

Normal distributions

Then,

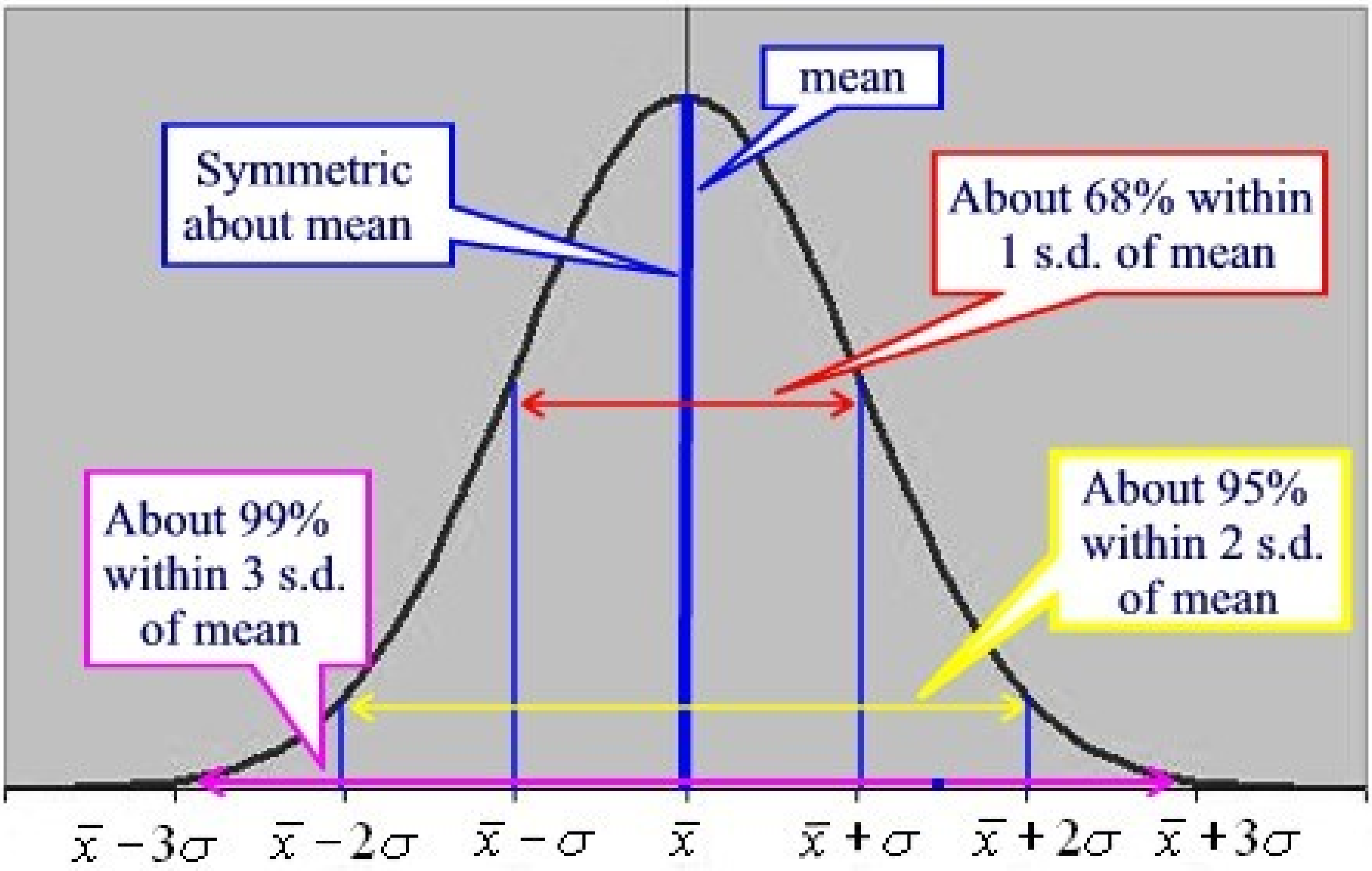
$$P(X > 140) \approx \int_{40}^{200} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/450} dx$$

$$\approx 0.0038$$

- About 0.4% of the population has an IQ over 140.

Normal distributions

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$



- Substitution method
- Integration by parts
- Improper integrals

Isaac Newton

Summarizing Exercise

a. Compute $\displaystyle\int_0^{\infty} \frac{x}{(1+x^2)^2}\ dx$

b. Compute $\displaystyle\int x \ln^2(x)\ dx$

Hint: apply partial integration twice

Mathematics B2: Newton

-Contents-

- [x] Integrals
- [x] Calculation techniques for integrals
- [ ] Power and Taylor series

- [ ] First order ODEs
- [ ] Complex numbers
- [ ] Second order ODEs