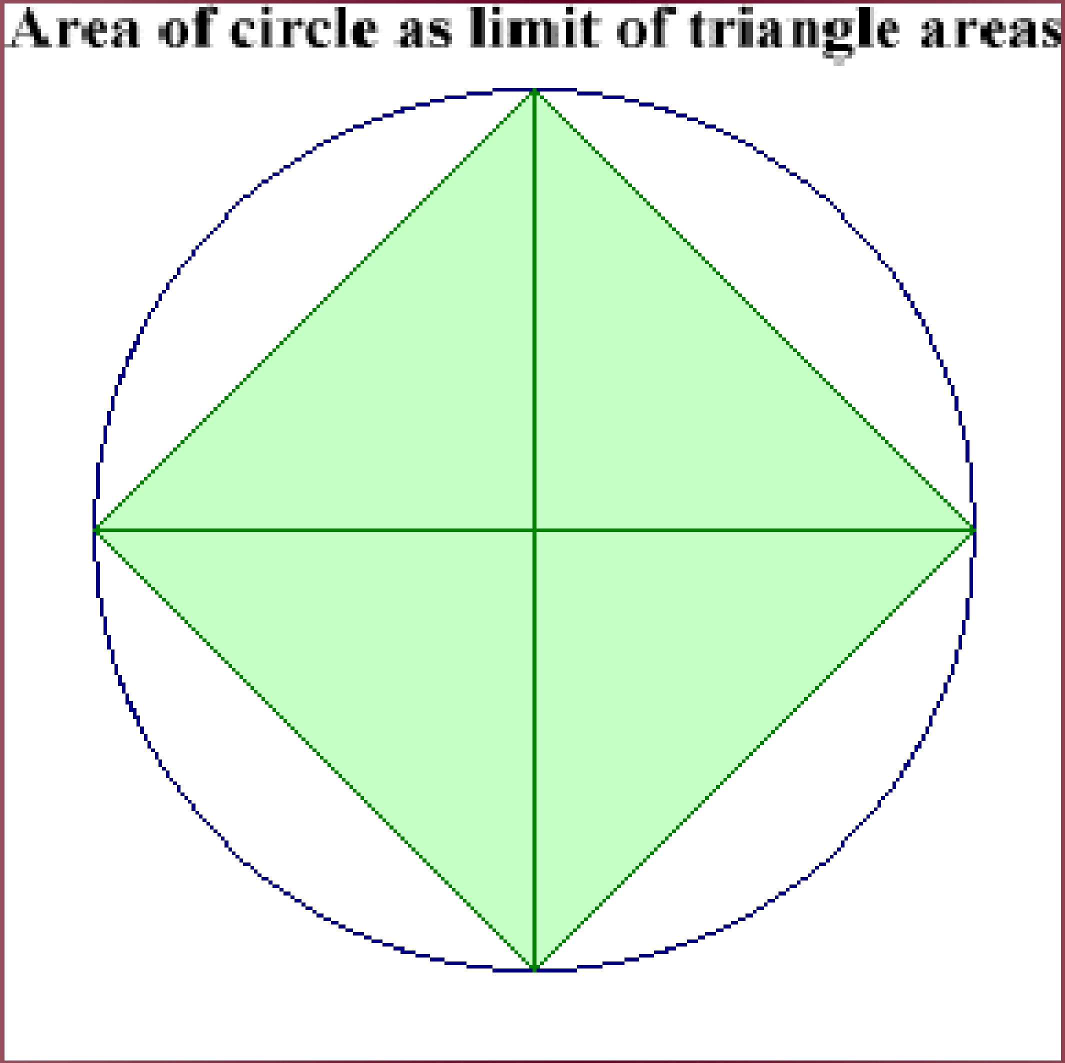
# Calculus 1B



# Calculus 1B

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Gerard Jeurnink

# Calculus 1A

functions, limits, differentiation

# Calculus 1B

## -Contents-

- Integrals
- Calculation techniques for integrals
- Power and Taylor series

- First order ODEs
- Complex numbers
- Second order ODEs

# Lecture 1 Integrals

- Theme: Area
- Theme: Riemann Sum
- Theme: Fundamental Theorem
- Theme: Antiderivatives

# Thomas' Calculus

# 5.1

## Area and Estimating with Finite Sums

# Area

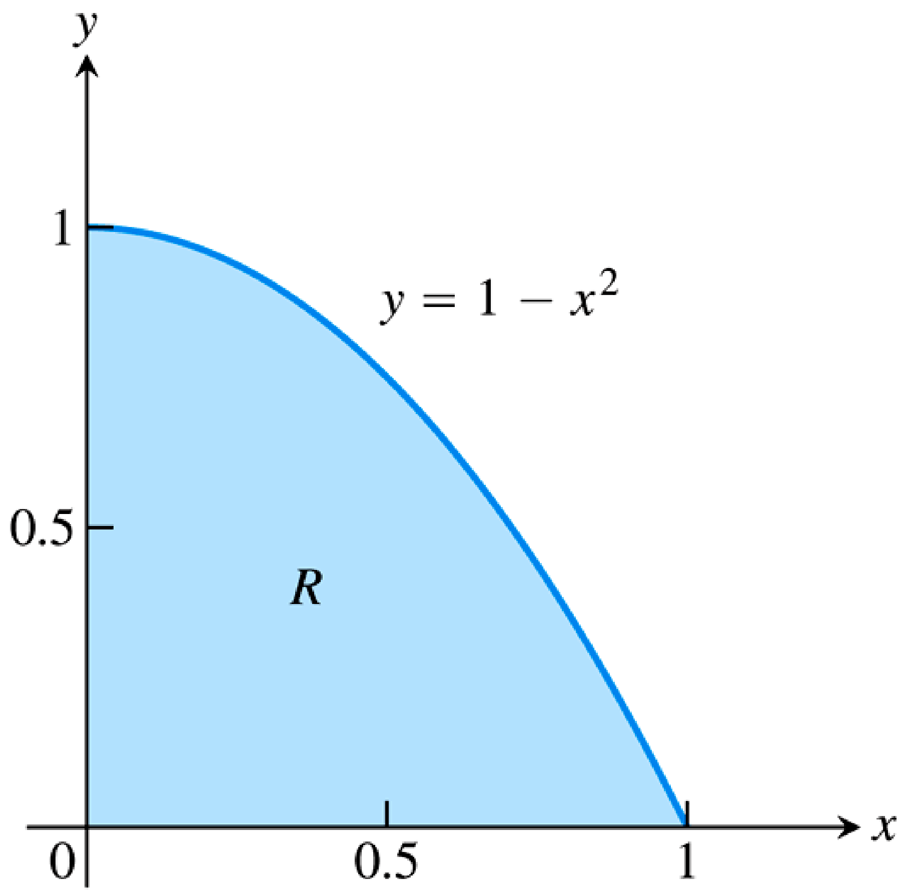



**FIGURE 5.1** The area of the region $R$ cannot be found by a simple formula.

# Area

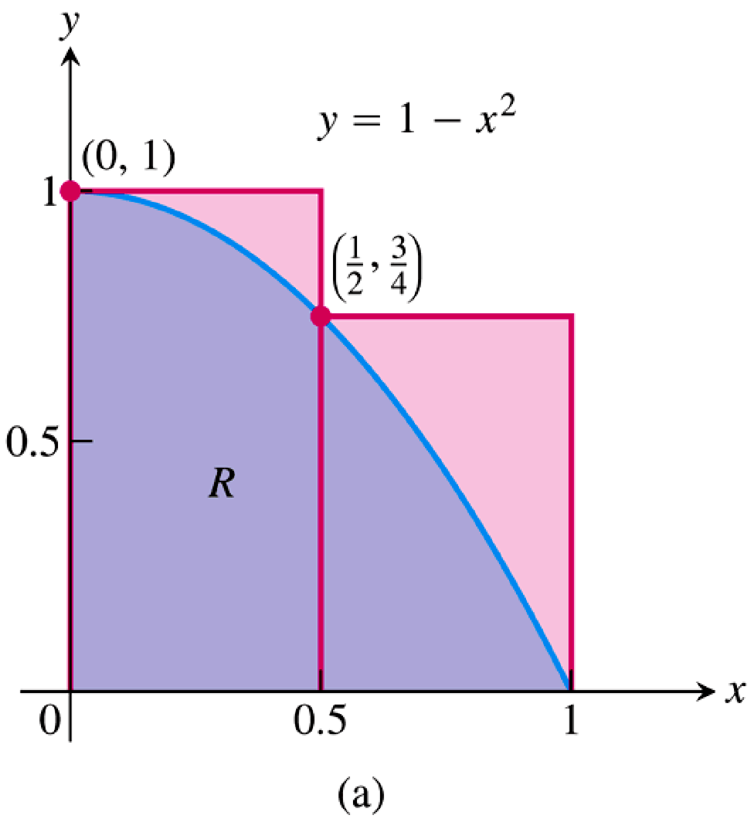



**FIGURE 5.2** (a) We get an upper estimate of the area of $R$ by using two rectangles containing $R$. (b) Four rectangles give a better upper estimate. Both estimates overshoot the true value for the area by the amount shaded in light red.

# Area

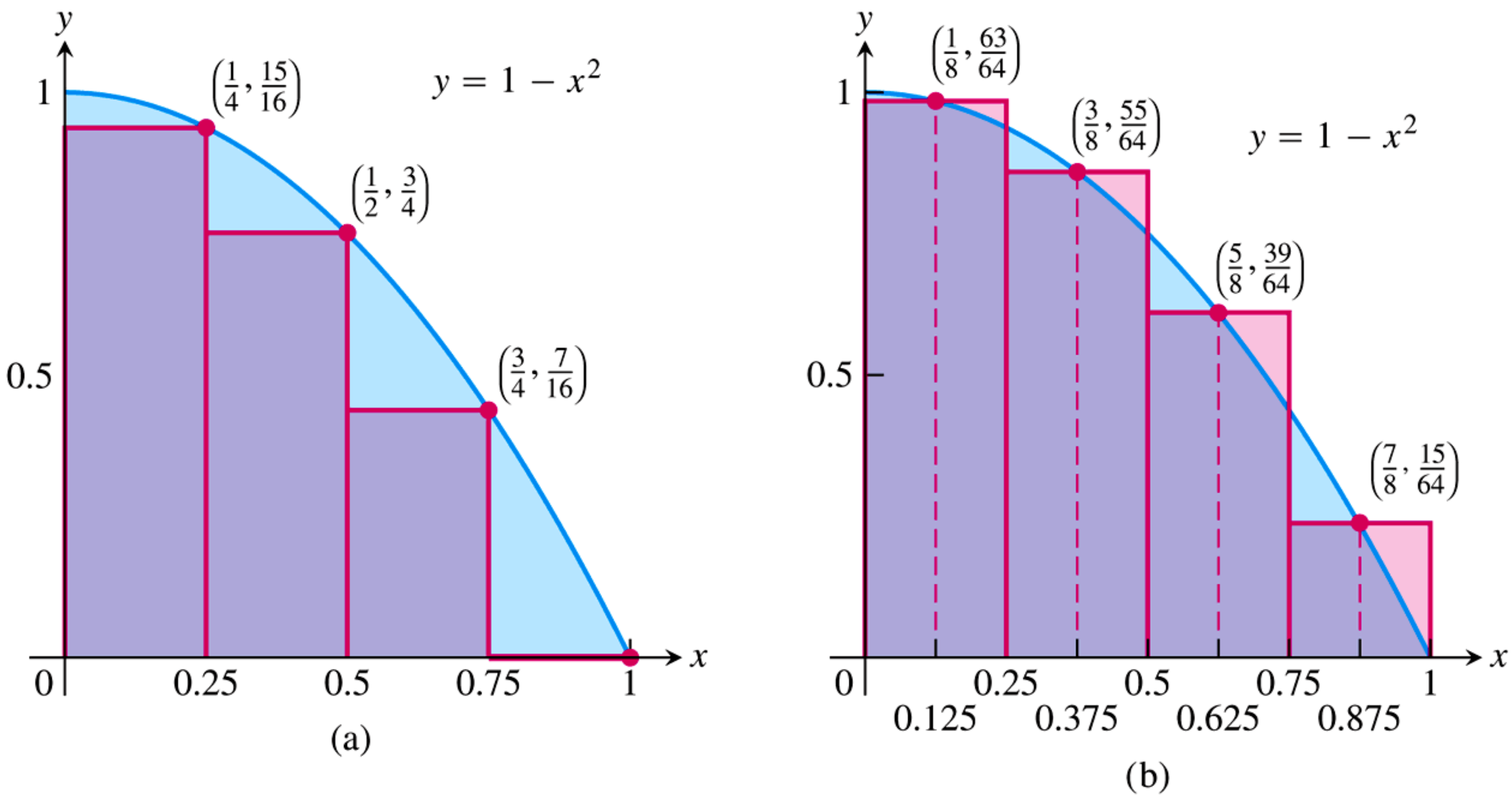



**FIGURE 5.3** (a) Rectangles contained in $R$ give an estimate for the area that undershoots the true value by the amount shaded in light blue. (b) The midpoint rule uses rectangles whose height is the value of $y = f(x)$ at the midpoints of their bases. The estimate appears closer to the true value of the area because the light red overshoot areas roughly balance the light blue undershoot areas.

# Area

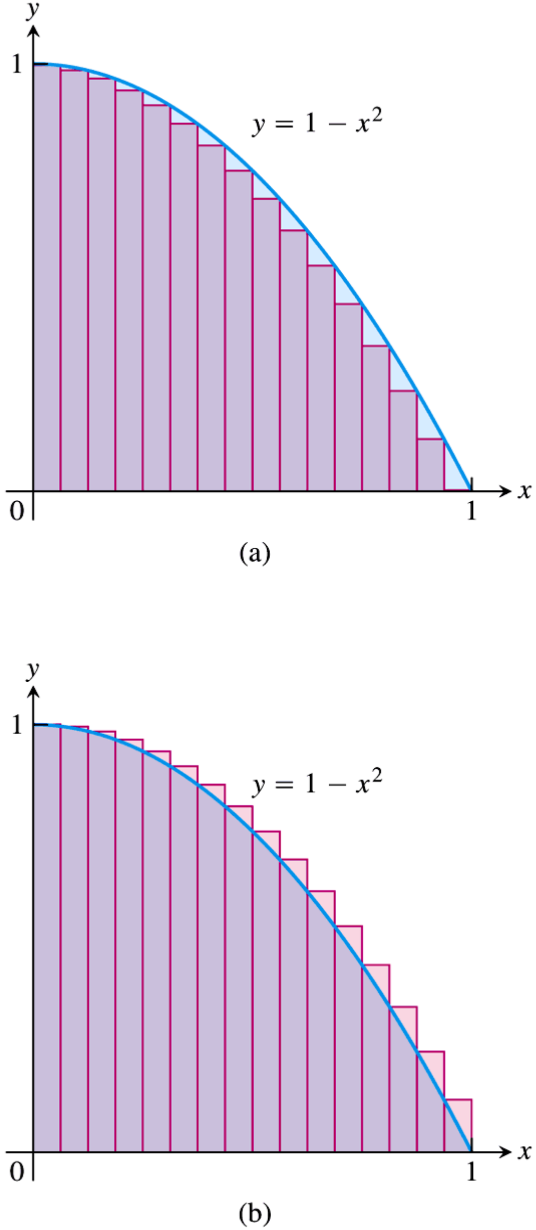



**FIGURE 5.4** (a) A lower sum using 16 rectangles of equal width $\Delta x = 1/16$. (b) An upper sum using 16 rectangles.

# Area

**TABLE 5.1** Finite approximations for the area of $R$

| Number of subintervals | Lower sum | Midpoint rule | Upper sum |
|---|---|---|---|
| 2 | .375 | .6875 | .875 |
| 4 | .53125 | .671875 | .78125 |
| 16 | .634765625 | .6669921875 | .697265625 |
| 50 | .6566 | .6667 | .6766 |
| 100 | .66165 | .666675 | .67165 |
| 1000 | .6661665 | .66666675 | .6671665 |

Upper and lower sums seem to converge to 0.6666… = 2/3

# 5.2

## Sigma Notation and Riemann Sums

# Σ-notation

The index $k$ ends at $k = n$.

$$\sum_{k=1}^{n} a_k$$

The summation symbol (Greek letter sigma)

$a_k$ is a formula for the $k$th term.

The index $k$ starts at $k = 1$.

# Σ-notation

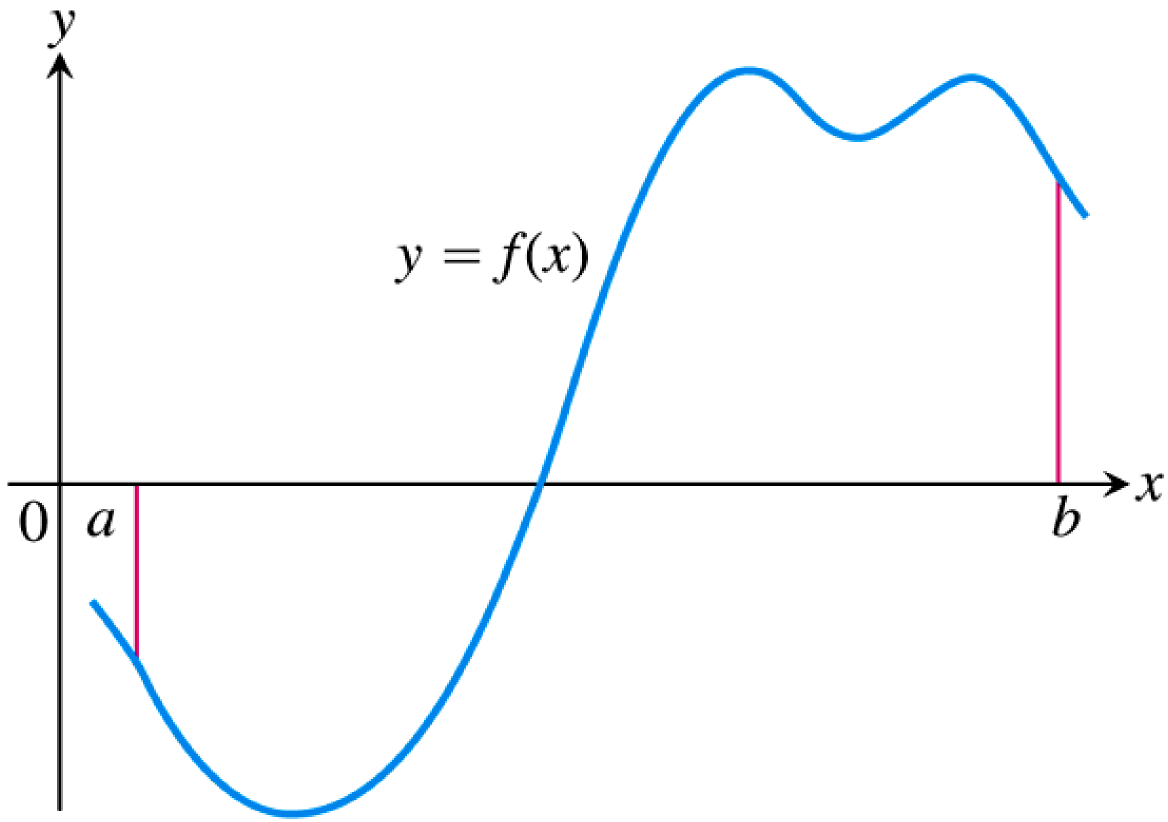
| The sum in sigma notation | The sum written out, one term for each value of $k$ | The value of the sum |
|---|---|---|
| $\sum_{k=1}^{5} k$ | $1 + 2 + 3 + 4 + 5$ | $15$ |
| $\sum_{k=1}^{3} (-1)^k k$ | $(-1)^1(1) + (-1)^2(2) + (-1)^3(3)$ | $-1 + 2 - 3 = -2$ |
| $\sum_{k=1}^{2} \frac{k}{k+1}$ | $\frac{1}{1+1} + \frac{2}{2+1}$ | $\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$ |
| $\sum_{k=4}^{5} \frac{k^2}{k-1}$ | $\frac{4^2}{4-1} + \frac{5^2}{5-1}$ | $\frac{16}{3} + \frac{25}{4} = \frac{139}{12}$ |

$$\sum_{k=0}^{1000} 1 = 1001$$

# Riemann sums

*Georg Friedrich Bernhard Riemann (1826-1866)*

# Riemann sums

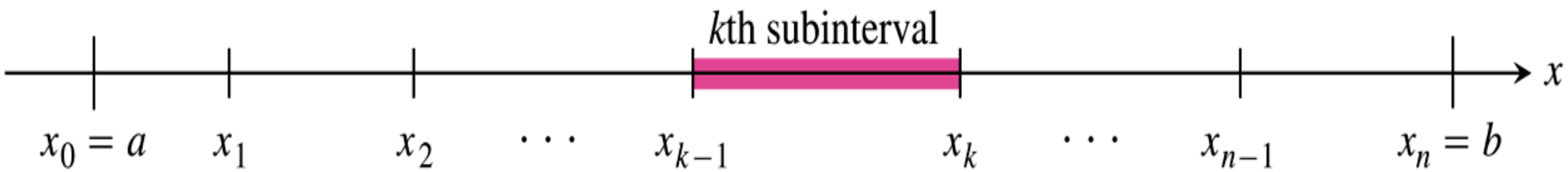



We want to approximate this "area", or integral.

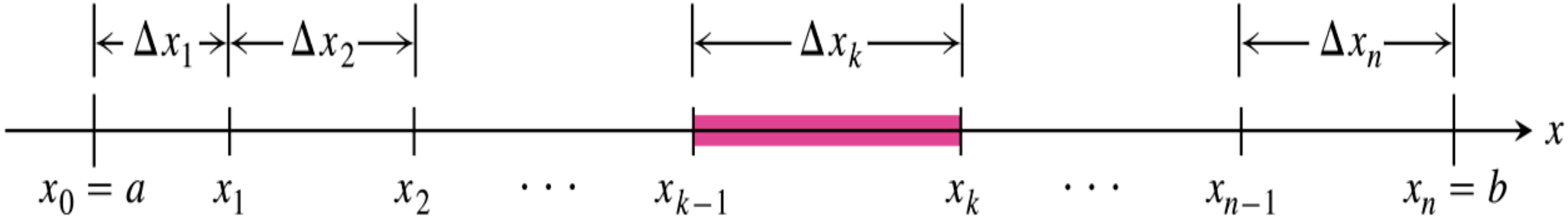
First divide $[a,b]$ in $n$ subintervals (*partition P*)

# Riemann sums

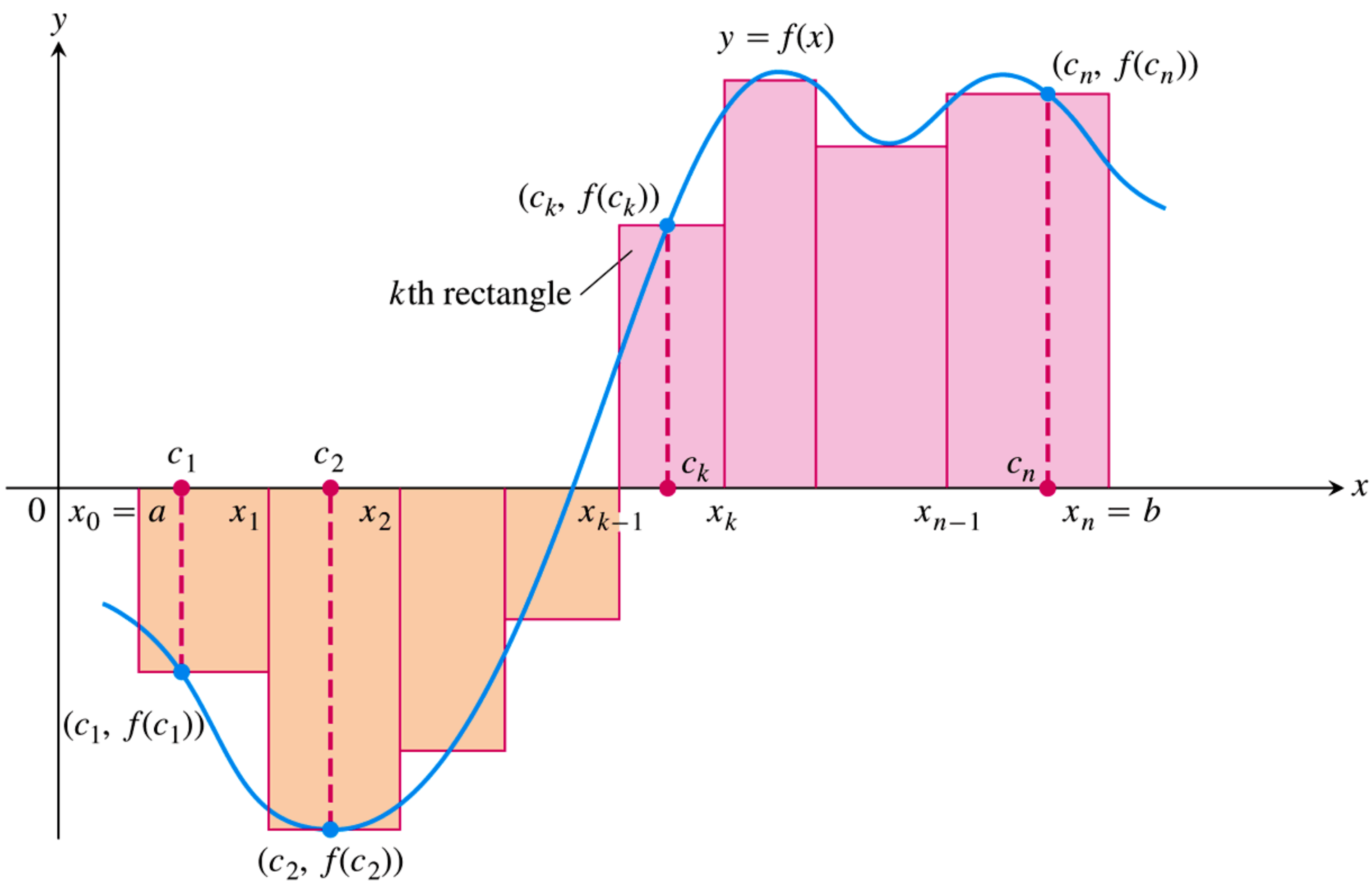
The $k^{th}$ subinterval of $P$ is $[x_{k-1}, x_k]$, for $k$ an integer between 1 and $n$




The width of the $k^{th}$ interval $\Delta x_k = x_k - x_{k-1}$, If all $n$ subintervals have equal width, then the common width $\Delta x_k$ is equal to $(b - a)/n$.

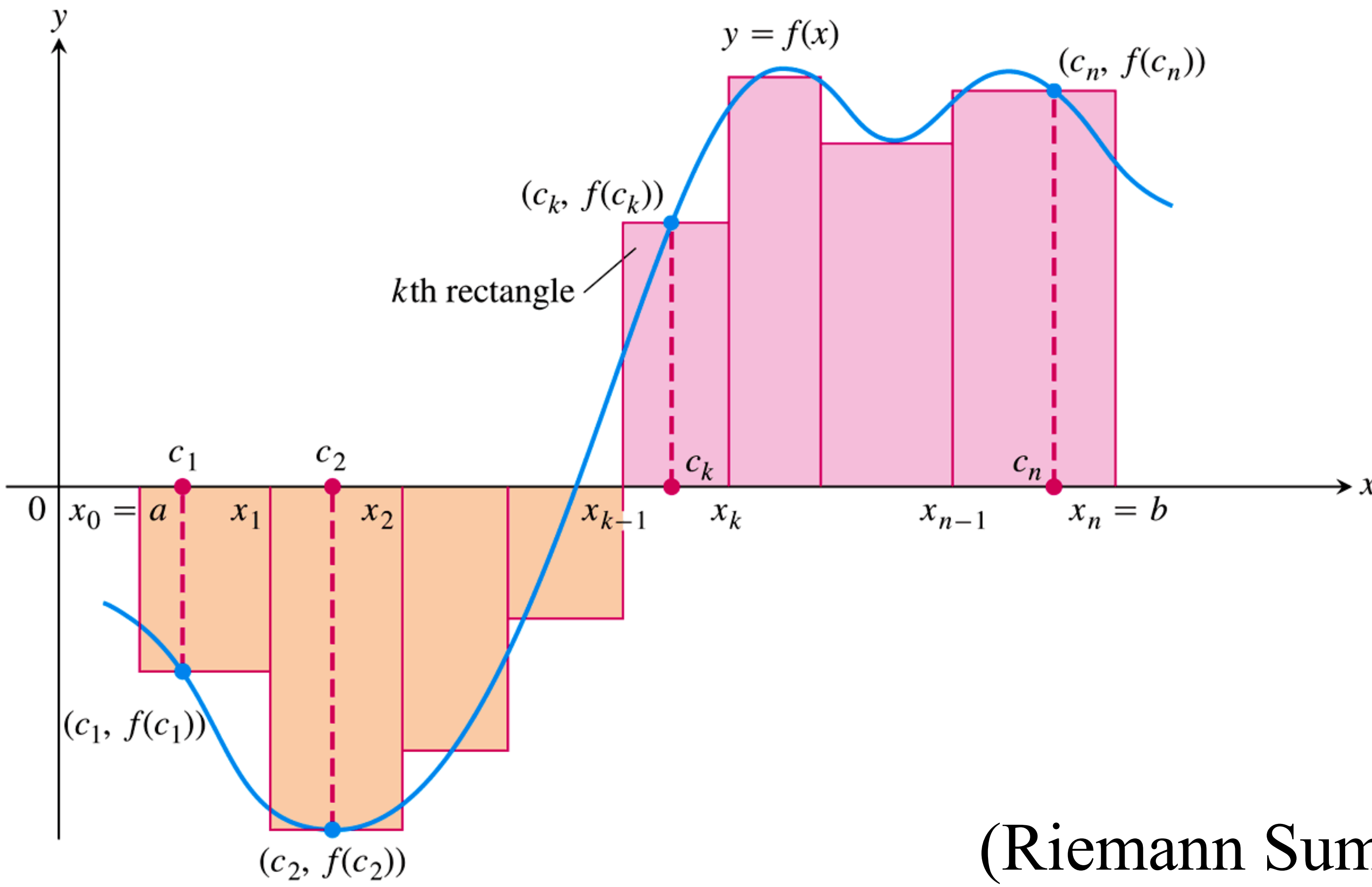


# Riemann sums




In each subinterval $[x_{k-1}, x_k]$ choose an element $c_k$ and erect the rectangle with area $f(c_k) \cdot (x_k - x_{k-1})$ …

# Riemann sums




(Riemann Sum)

$$\text{Integral} \approx \sum_{k=1}^{n} f(c_k)\cdot(x_k - x_{k-1}) = \sum_{k=1}^{n} f(c_k)\cdot\Delta x_k$$

# Riemann sums

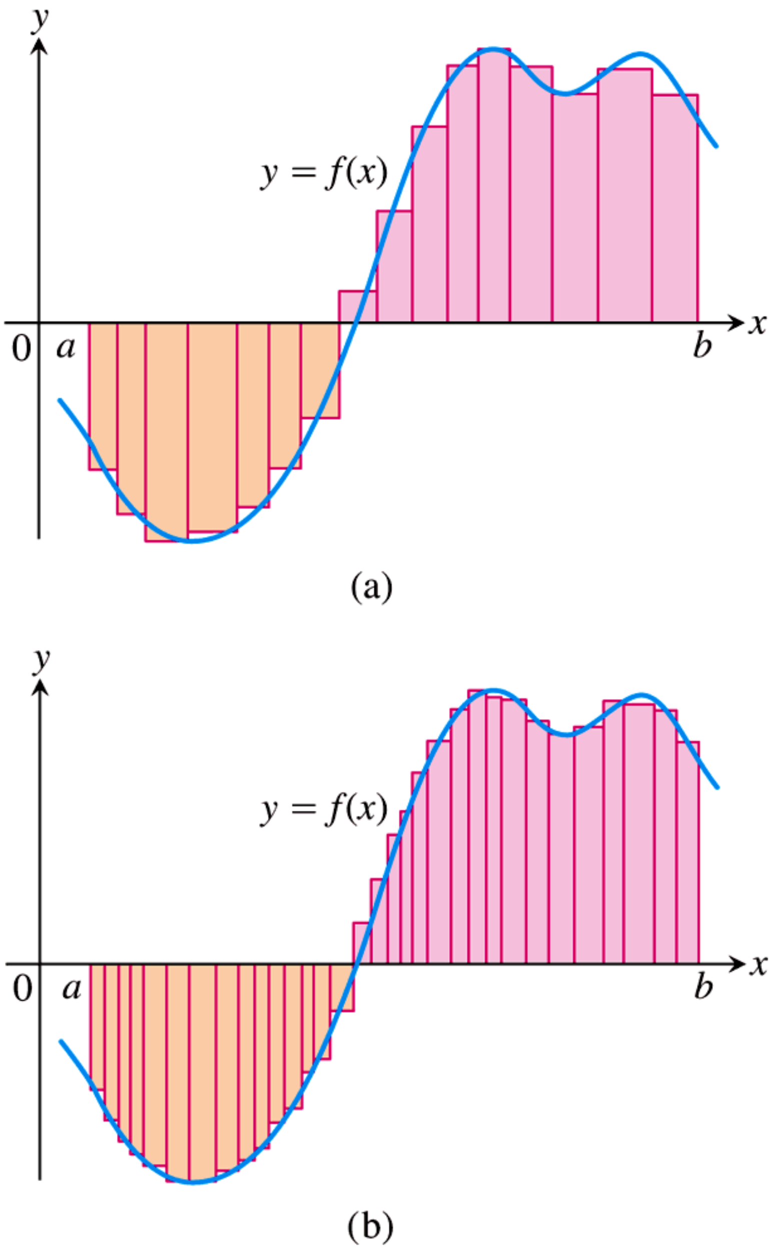



**FIGURE 5.10** The curve of Figure 5.9 with rectangles from finer partitions of $[a, b]$. Finer partitions create collections of rectangles with thinner bases that approximate the region between the graph of $f$ and the $x$-axis with increasing accuracy.

# 5.3

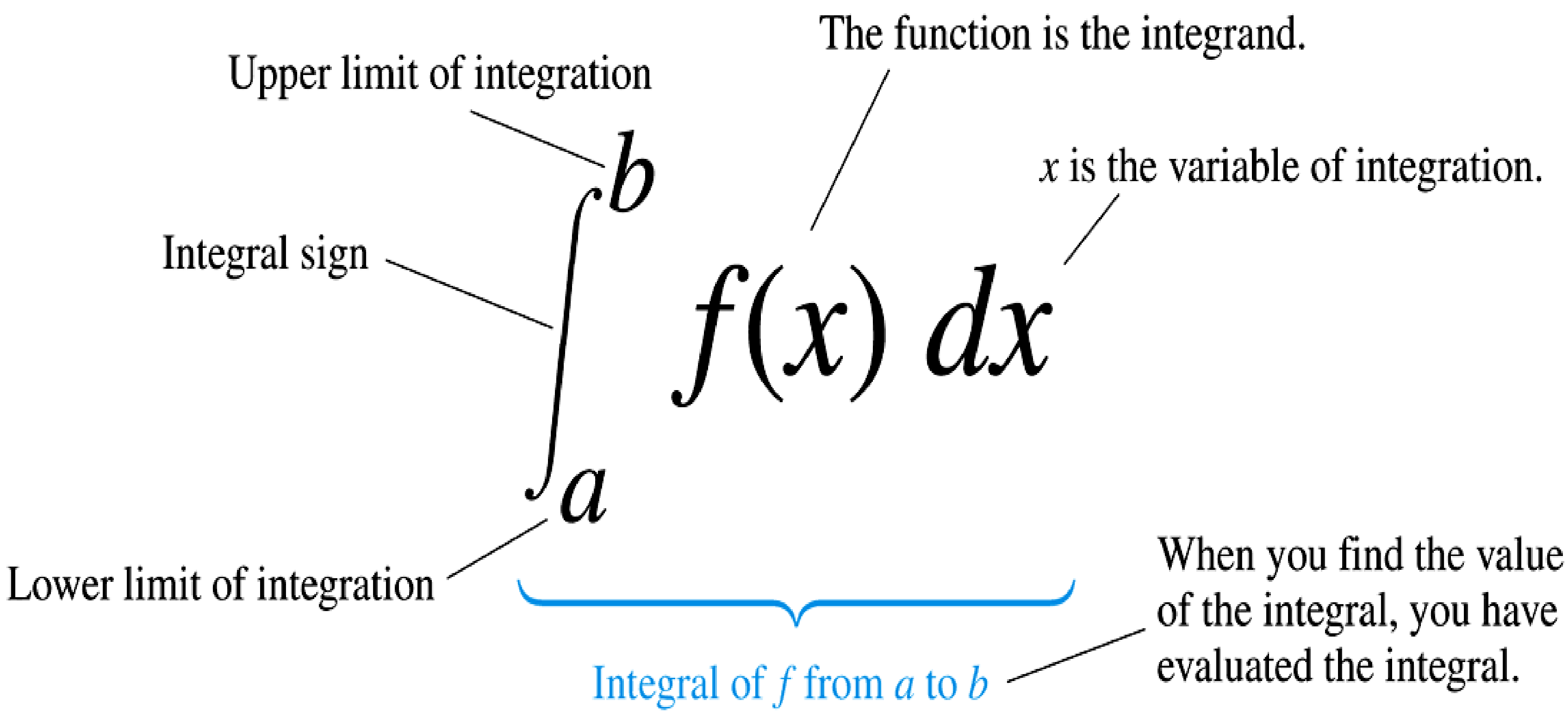
## The Definite Integral

# Definite Integral

$$\int_a^b f(x)dx = \underset{n\to\infty}{\text{"lim"}}\left(\sum_{i=1}^{n} f(t_i)\cdot\Delta x_i\right)$$

**The definite integral exists if all Riemann sums converges to the same number.**

# Definite Integral



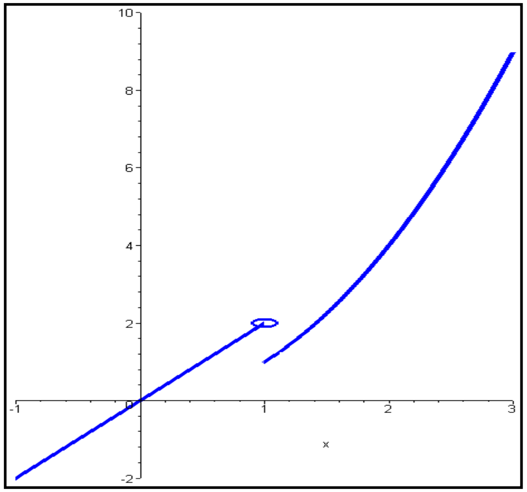
# integral $\neq$ area

$$\text{an integral } \int = \text{limit of Riemann sums}$$

# Definite Integral

**THEOREM 1—Integrability of Continuous Functions** If a function $f$ is continuous over the interval $[a, b]$, or if $f$ has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)\,dx$ exists and $f$ is integrable over $[a, b]$.

A jump discontinuity in *a* means that *f* is *not* continuous, but the left-hand and right-hand limits of *f* in *a* exist.




A jump discontinuity

# Rules for integrals

# Rules for integrals

**TABLE 5.4** Rules satisfied by definite integrals

| | | | |
|---|---|---|---|
| **1.** | *Order of Integration:* | $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ | A Definition |
| **2.** | *Zero Width Interval:* | $\int_a^a f(x)\,dx = 0$ | A Definition when $f(a)$ exists |
| **3.** | *Constant Multiple:* | $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ | Any constant $k$ |
| **4.** | *Sum and Difference:* | $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$ | |
| **5.** | *Additivity:* | $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$ | |
| **6.** | *Max-Min Inequality:* | If $f$ has maximum value max $f$ and minimum value min $f$ on $[a, b]$, then $\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a).$ | |
| **7.** | *Domination:* | $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$ | |
| | | $f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$ | (Special Case) |

Area, mean, volume, …

# Area and mean

**DEFINITION** If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of $f$ from $a$ to $b$,

$$A = \int_a^b f(x)\,dx.$$

**DEFINITION** If $f$ is integrable on $[a, b]$, then its **average value on $[a, b]$**, also called its **mean**, is

$$\text{av}(f) = \frac{1}{b - a} \int_a^b f(x)\,dx.$$

# Area and mean

**Example:**

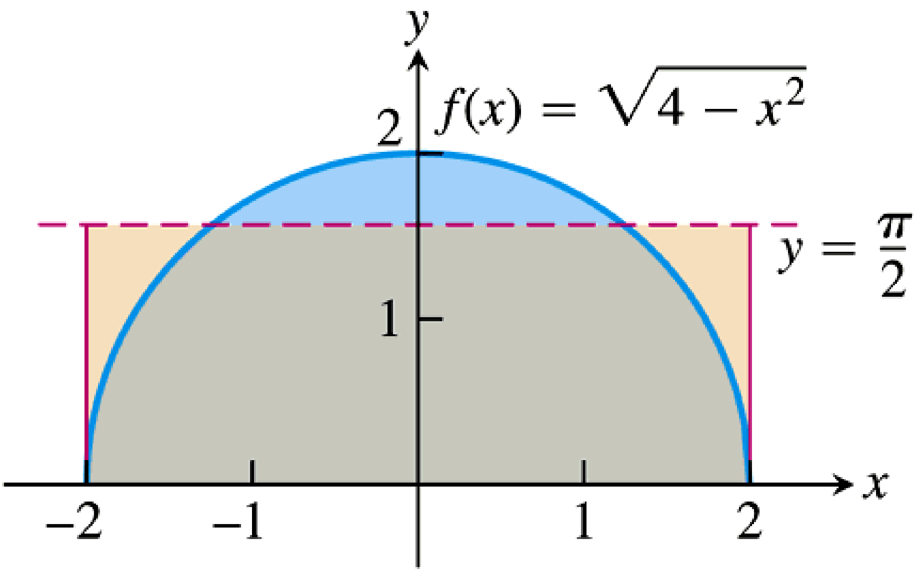



**FIGURE 5.15** The average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ is $\pi/2$

Because: the area of a half circle with radius 2 is 2π and the width ($b - a$) is 4, leading to mean ½ π.

# Area and mean and ...

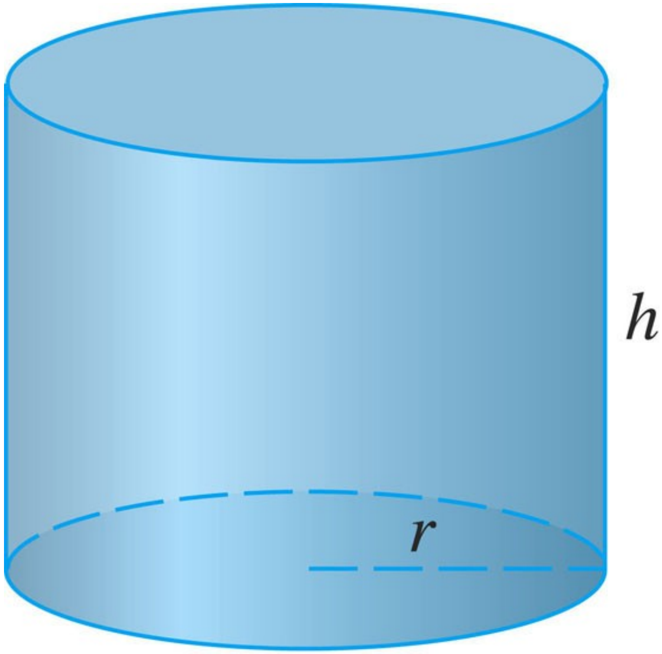
$$\text{an integral } \int = \text{limit of Riemann sums}$$

# Volume

# Example:

If the base is a circle with radius *r*, then circular cylinder with height *h* has volume:

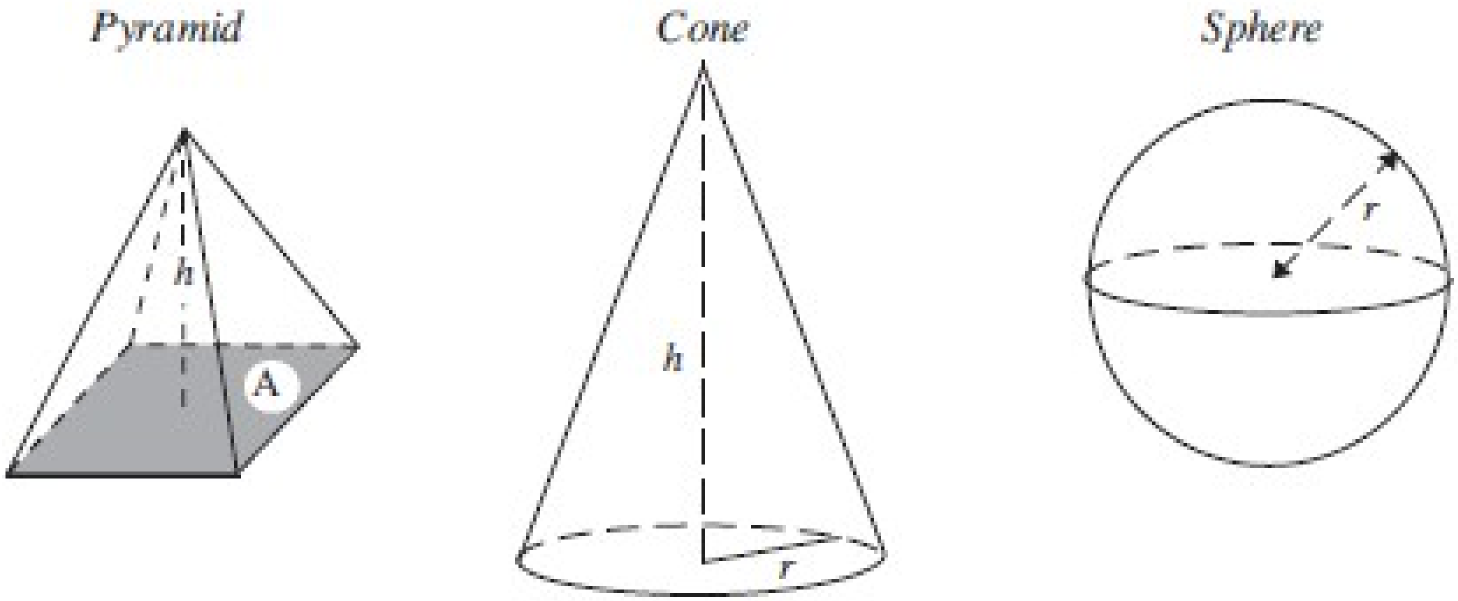
$$V = \pi r^2 h$$




Circular cylinder
$V = \pi r^2 h$

# Volume

## Pyramid Cone Sphere

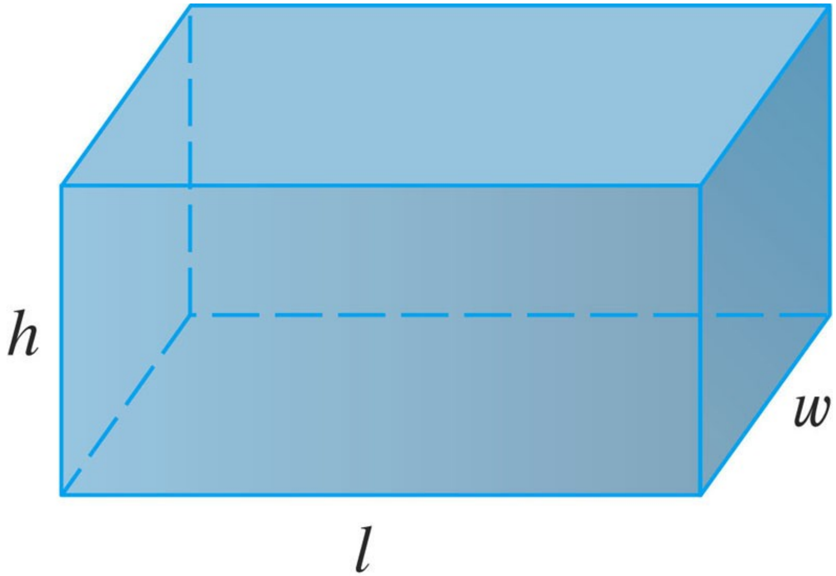



$$V = \frac{1}{3}Ah \qquad V = \frac{1}{3}\pi r^2 h \qquad V = \frac{4}{3}\pi r^3$$

## Example:

If the base is a rectangle with length $l$ and width $w$ then the rectangular parallelepiped with height $h$ has volume:

$$V = lwh$$

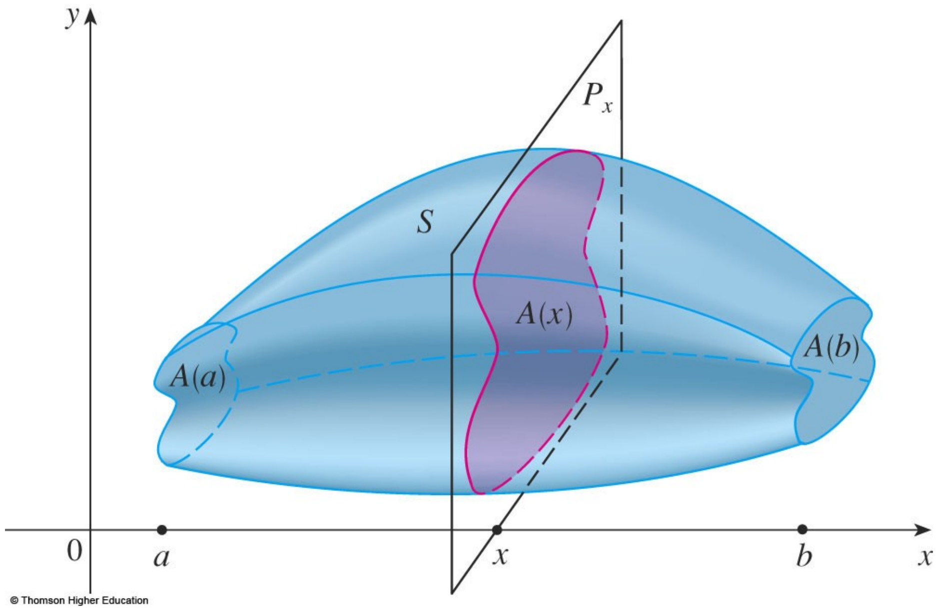



Rectangular box
$V = lwh$

# Volume by cross-sections

# Volume by cross-sections

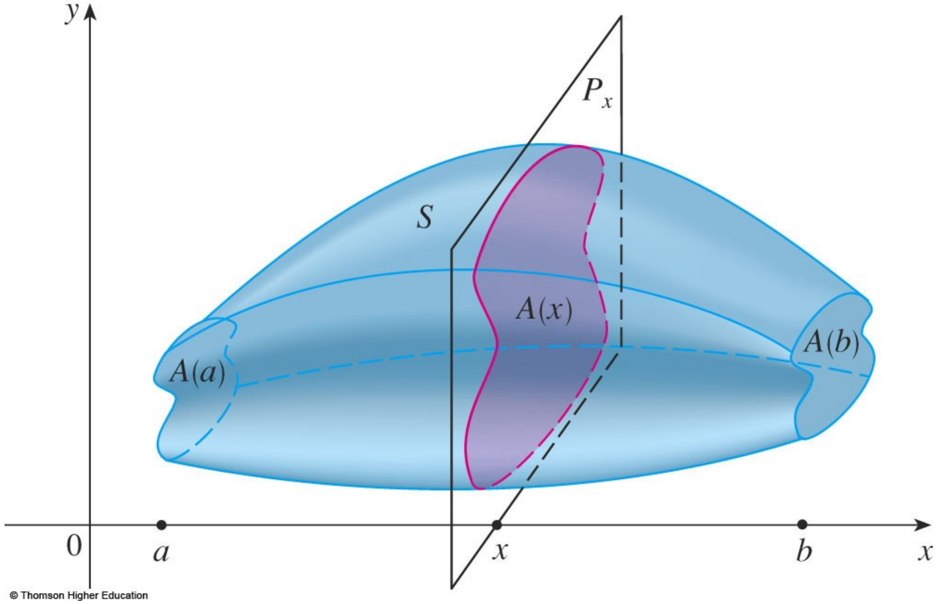
We start by intersecting *S* with a plane and obtaining a plan region that is called a *cross-section* of *S*



# Volume by cross-sections

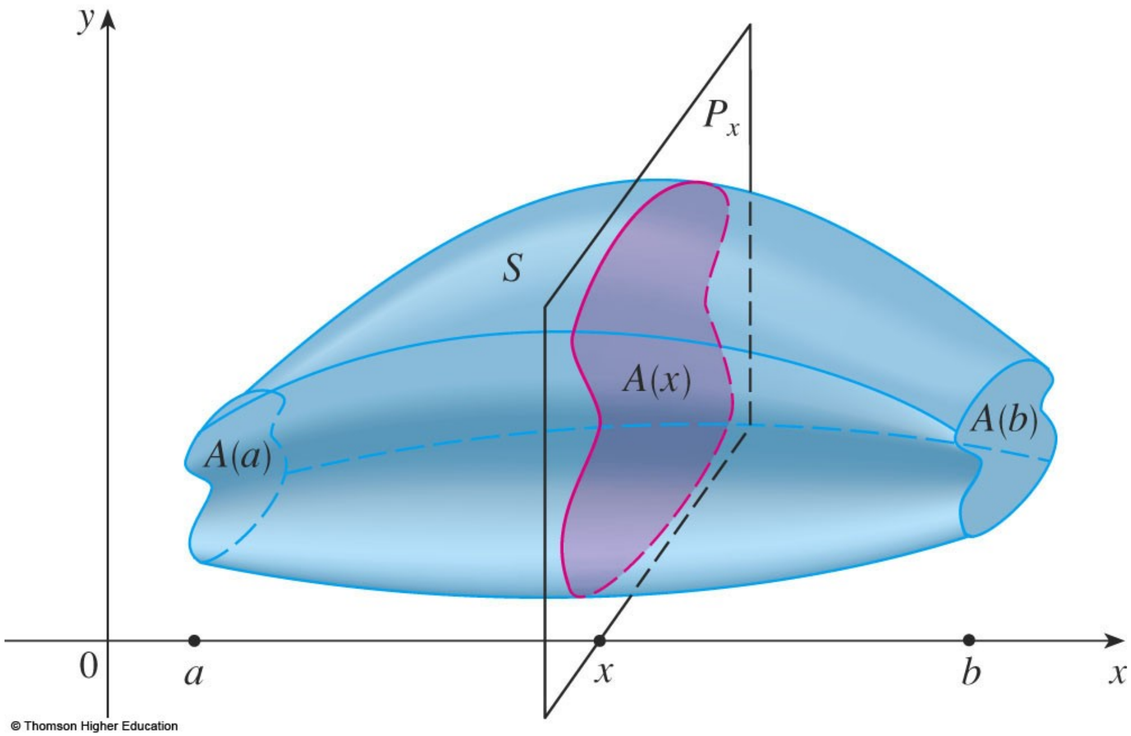
Let $A(x)$ be the area of the cross-section of $S$ in a plane $P_x$ perpendicular to the $x$-axis and passing through the point $x$, where $a \le x \le b$.

Think of slicing $S$ with a knife through $x$ and computing the area of this slice.



# Volume by cross-sections

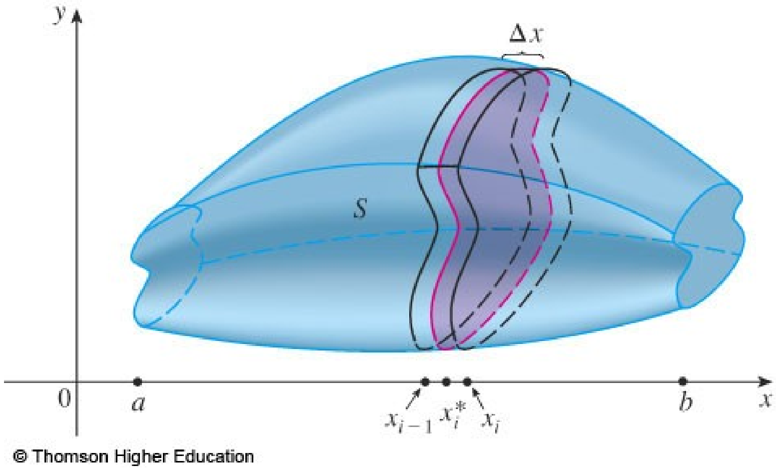
The cross-sectional area *A(x)* will vary as *x* increases from *a* to *b*.

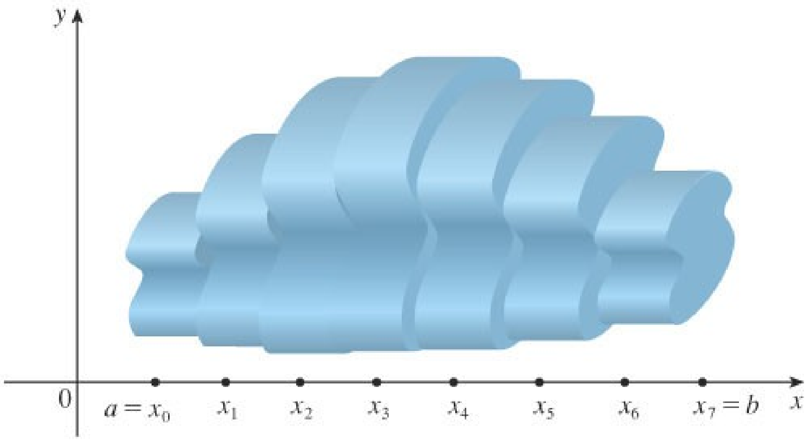


# Volume by cross-sections

We divide $S$ into $n$ 'slabs' of equal width $\Delta x$ using the planes $P_{x1}$, $P_{x2}$, ... to slice the solid

Think of slicing a loaf of bread



# Volume by cross-sections

So, an approximation to our intuitive conception of the volume of the *i*-th slab $S_i$ is:

$$V(S_i) \approx A(x_i)\Delta x$$

# Volume by cross-sections

Adding the volumes of these slabs, we get an approximation to the total volume (that is, what we think of intuitively as the volume):

$$V \approx \sum_{i=1}^{n} A(x_i)\Delta x$$

- This approximation appears to become better and better as $n \to \infty$
- Think of the slices as becoming thinner and thinner

# Volume by cross-sections

$$V \approx \sum_{i=1}^{n} A(x_i)\Delta x$$

Therefore, we define the volume as the limit of these sums as $n \longrightarrow \infty$.

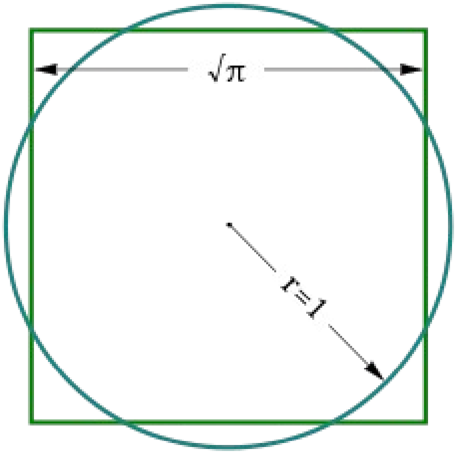
However, we recognize the limit of Riemann sums as a definite integral and so we have the following definition.

## Definition of Volume:

Let $S$ be a solid that lies between $x = a$ and $x = b$.
If the cross-sectional area of $S$ in the plane $P_x$ through $x$ and perpendicular to the x-axis, is $A(x)$ then the volume of $S$ is:

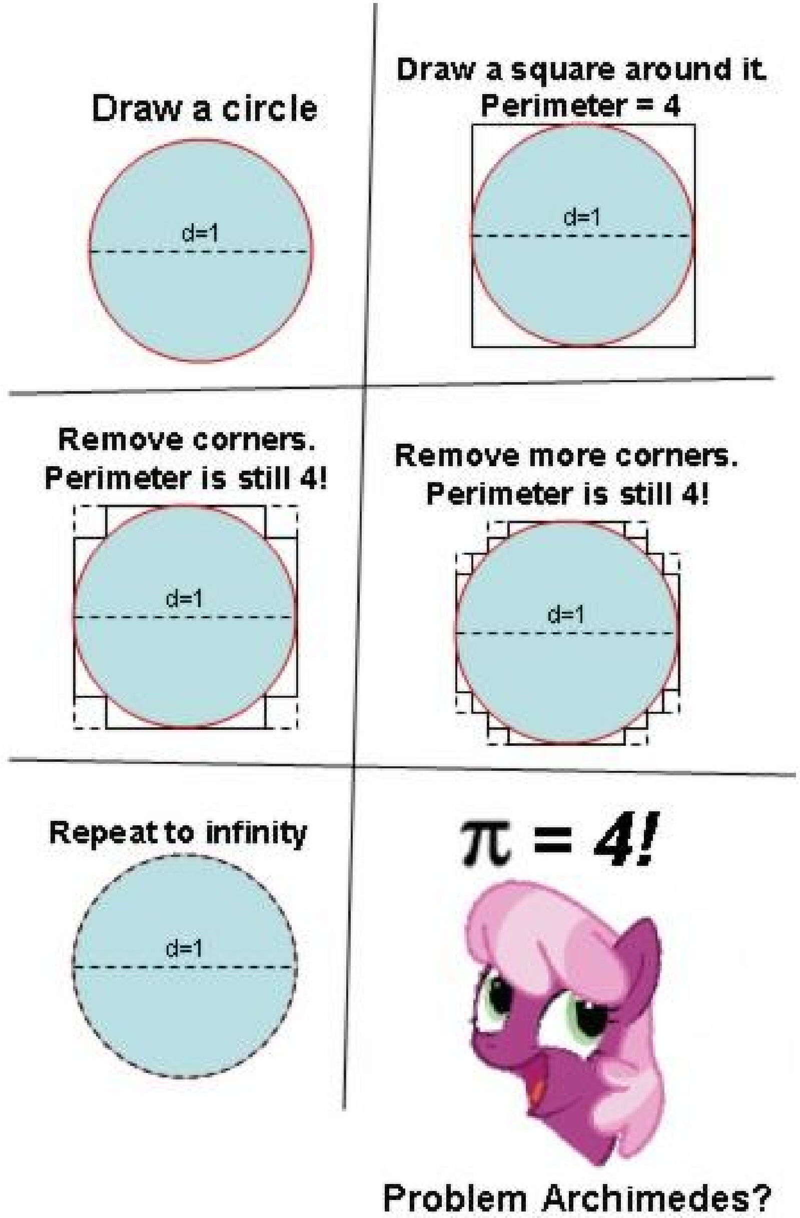
$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i) \Delta x = \int_a^b A(x) dx$$

# Squaring the circle




Squaring the circle: the areas of this square and this circle are both equal to π. In 1882, it was proven that this figure cannot be constructed in a finite number of steps with an idealized compass and straightedge.

# Break


Draw a circle
d=1
Draw a square around it
Perimeter = 4
d=1
Remove corners.
Perimeter is still 4!
d=1
Remove more corners.
Perimeter is still 4!
d=1
Repeat to infinity
d=1
π = 4!
Problem Archimedes?

# 5.4

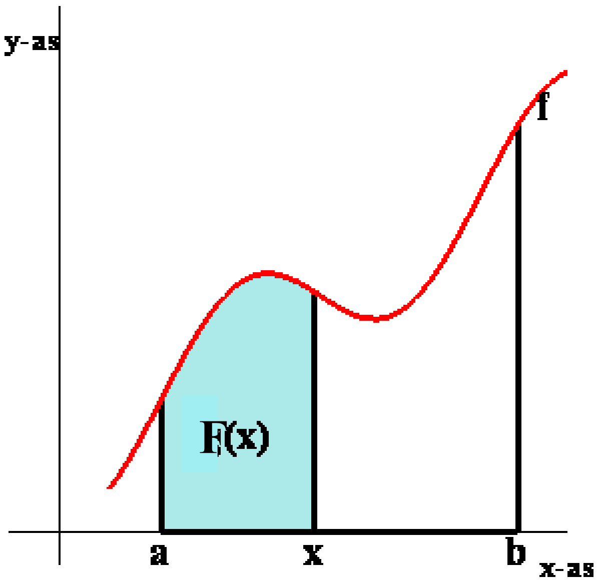
# The Fundamental Theorem of Calculus

# Fundamental Theorem of Calculus

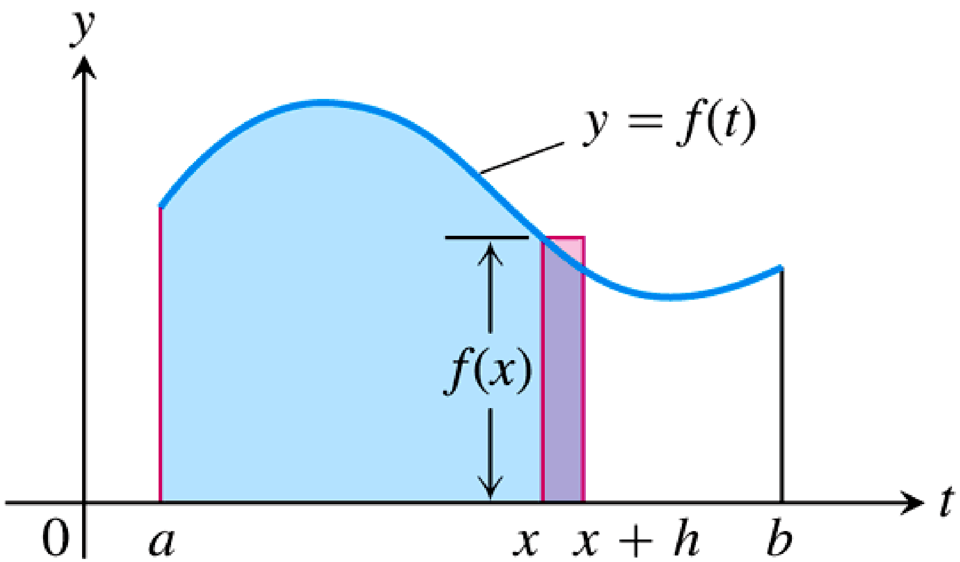
Given is a continuous function *f* on the interval [*a*, *b*].
Define the function *F* by:

$$F(x) = \int_a^x f(t)\,dt$$

The interpretation of *F* is:
the area under the graph of *f*
on the interval from *a* to *x*.



# Fundamental Theorem of Calculus




We want to find $F'(x)$:

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

# The Fundamental Theorem of Calculus, Part 1

**THEOREM 4—The Fundamental Theorem of Calculus, Part 1** If $f$ is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on $(a, b)$ and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

# Fundamental Theorem of Calculus ‘proof’

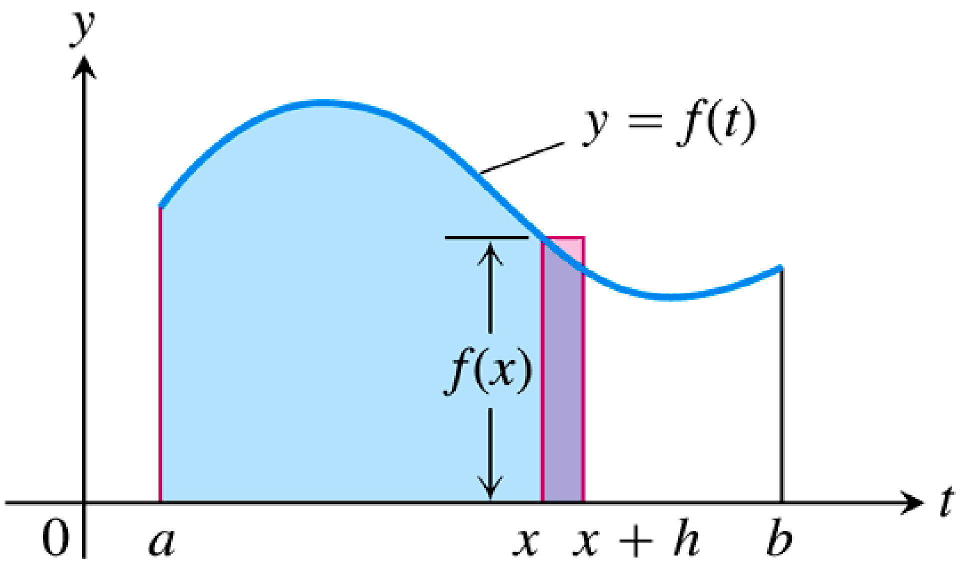



**FIGURE 5.19** In Equation (1), $F(x)$ is the area to the left of $x$. Also, $F(x + h)$ is the area to the left of $x + h$. The difference quotient $[F(x + h) - F(x)]/h$ is then approximately equal to $f(x)$, the height of the rectangle shown here.

# Fundamental Theorem of Calculus

**THEOREM** **The Fundamental Theorem of Calculus, Part 2** If $f$ is continuous at every point in $[a, b]$ and $F$ is any antiderivative of $f$ on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This result is independent of your choice of an *antiderivative* of $f$. If $G(x)$ is another antiderivative, i.e. also $G' = f$, then

$$F(x) = G(x) + C$$

But then:

$$F(b) - F(a) = (G(b) + C) - (G(a) + C) = G(b) - G(a)$$

# Fundamental Theorem of Calculus

A function with derivative $f$ is called an *antiderivative* of $f$. We see that the integral function:

$$F(x) = \int_a^x f(t)dt$$

is an antiderivative of $f$, independent of the starting point $a$.

# Fundamental Theorem of Calculus

**THEOREM** **The Net Change Theorem** The net change in a function $F(x)$ over an interval $a \le x \le b$ is the integral of its rate of change:

$$F(b) - F(a) = \int_a^b F'(x)\, dx.$$

Remember the differential: $dF = F'(x)dx$

Then: $$F(b) - F(a) = \int_a^b dF$$

# 4.8

## Antiderivatives

# Anti-derivatives

**DEFINITION** A function $F$ is an **antiderivative** of $f$ on an interval $I$ if $F'(x) = f(x)$ for all $x$ in $I$.

**THEOREM 8** If $F$ is an antiderivative of $f$ on an interval $I$, then the most general antiderivative of $f$ on $I$ is

$$F(x) + C$$

where $C$ is an arbitrary constant.

# Anti-derivatives

**TABLE 4.2** Antiderivative formulas, $k$ a nonzero constant

| | Function | General antiderivative | | Function | General antiderivative |
|---|---|---|---|---|---|
| **1.** | $x^n$ | $\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$ | **8.** | $e^{kx}$ | $\frac{1}{k}e^{kx} + C$ |
| **2.** | $\sin kx$ | $-\frac{1}{k}\cos kx + C$ | **9.** | $\frac{1}{x}$ | $\ln\lvert x\rvert + C, \quad x \neq 0$ |
| **3.** | $\cos kx$ | $\frac{1}{k}\sin kx + C$ | **10.** | $\frac{1}{\sqrt{1-k^2x^2}}$ | $\frac{1}{k}\sin^{-1} kx + C$ |
| **4.** | $\sec^2 kx$ | $\frac{1}{k}\tan kx + C$ | **11.** | $\frac{1}{1+k^2x^2}$ | $\frac{1}{k}\tan^{-1} kx + C$ |
| **5.** | $\csc^2 kx$ | $-\frac{1}{k}\cot kx + C$ | **12.** | $\frac{1}{x\sqrt{k^2x^2-1}}$ | $\sec^{-1} kx + C, \quad kx > 1$ |
| **6.** | $\sec kx \tan kx$ | $\frac{1}{k}\sec kx + C$ | **13.** | $a^{kx}$ | $\left(\frac{1}{k \ln a}\right)a^{kx} + C, \quad a > 0, a \neq 1$ |
| **7.** | $\csc kx \cot kx$ | $-\frac{1}{k}\csc kx + C$ | | | |

# Variations on the Fundamental Theorem

# Variations on the Fundamental Theorem

Define: $$g(x) = \int_x^b f(t)\,dt$$

where *f* is continuous is on [*a*, *b*] . Then *g* is differentiable on (*a*, *b*), and

$$g'(x) = -f(x)$$

Indeed: $$g(x) = -\int_b^x f(t)\,dt$$

# Variations on the Fundamental Theorem

**Examples:**

$$g(x) = \int_2^x \sqrt{1+t^4}\,dt \quad \rightarrow \quad g'(x) = \sqrt{1+x^4}$$

$$h(s) = \int_0^s \sqrt{3+\sin x}\,dx \quad \rightarrow \quad h'(s) = \sqrt{3+\sin s}$$

$$k(t) = \int_t^0 \ln\left(1+s^2\right)ds \quad \rightarrow \quad k'(t) = -\ln\left(1+t^2\right)$$

# Variations on the Fundamental Theorem

**Example:** $$f(x) = \int_0^{x^2} \sqrt{1+t}dt$$

Then $f(x) = g(x^2)$, where

$$g(u) = \int_0^{u} \sqrt{1+t}dt$$

Then $$f'(x) = g'(x^2) \cdot 2x = \sqrt{1+x^2} \cdot 2x$$

# Quiz

Given $$h(x) = \int_x^{x^2} \tan(t)dt$$

Then $h'(x)$ equals

a) $\tan(x^2)$

b) $\tan(x^2) - \tan(x)$

c) $2x\tan(x^2) - \tan(x)$

d) $2x\tan(x^2) - 2x\tan(x)$

# Quiz

Given $$h(x) = \int_{x}^{x^2} \tan(t)dt$$

Then $h'(x)$ equals

a)
b)
c) $2x\tan(x^2) - \tan(x)$
d)

# Indefinite integrals

# Indefinite integrals

**DEFINITION** The collection of all antiderivatives of $f$ is called the **indefinite integral** of $f$ with respect to $x$, and is denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign**. The function $f$ is the **integrand** of the integral, and $x$ is the **variable of integration**.

# Indefinite integrals

**Examples:**

$$\int\left(\frac{3}{x}-\frac{2}{1+x^2}\right)dx \quad = 3\ln|x| - 2\tan^{-1}x + C$$

Check by differentiation!

$$\int\left(2^x - 2\sqrt{x}\right)dx \quad = \int\left(2^x - 2x^{\frac{1}{2}}\right)dx \quad = \frac{2^x}{\ln 2} - \frac{4}{3}x^{\frac{3}{2}} + C \quad = \frac{2^x}{\ln 2} - \frac{4}{3}x\sqrt{x} + C$$

Check by differentiation!

$$\int e^{x^2}\,dx = ???$$

It exists, but is not expressible in known functions.

## The Gini index is an integral

# The Gini index is an integral

How can we measure the distribution of income among the inhabitants of a country?

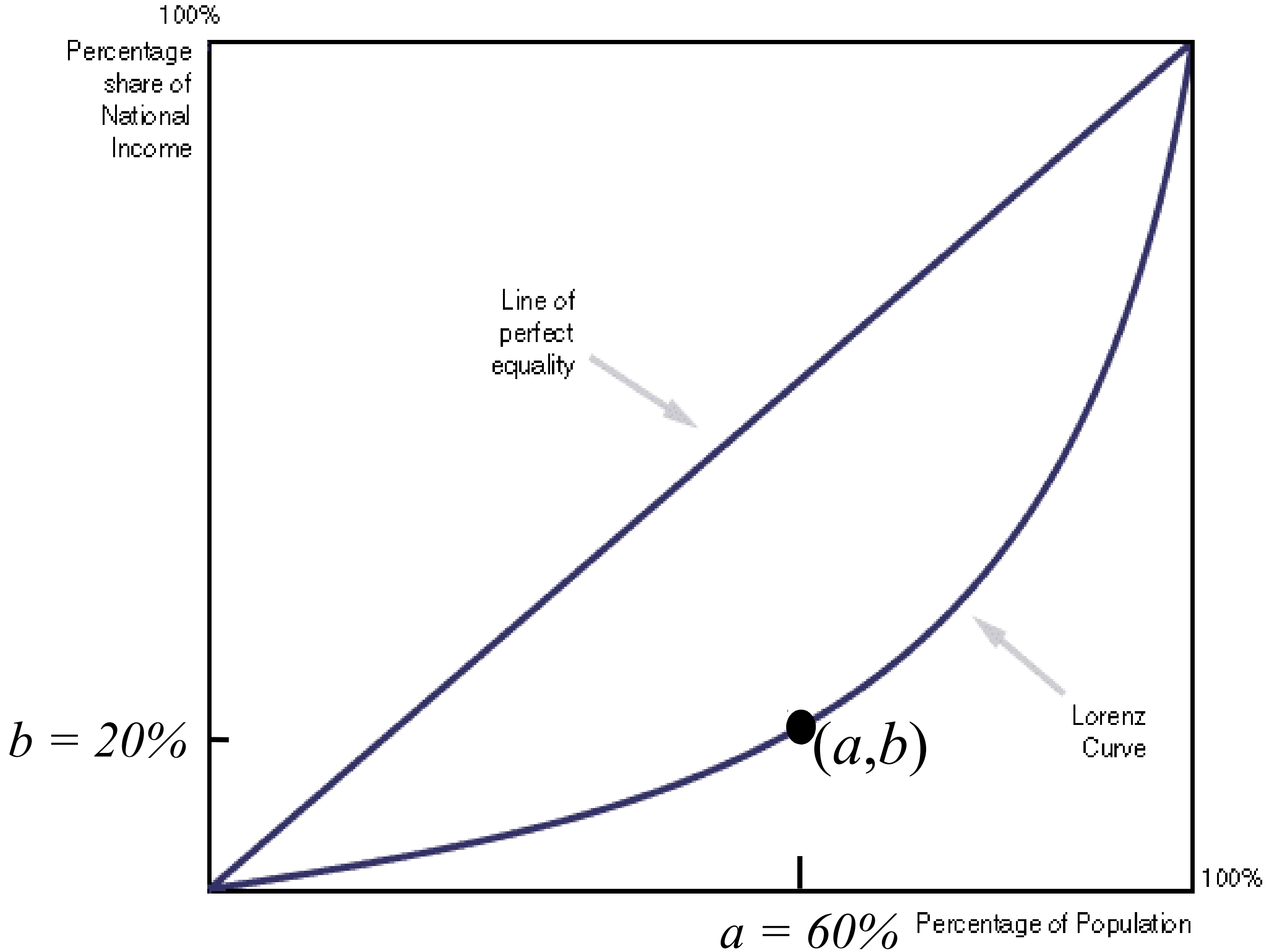
Corrado Gini (1884 – 1965)

# The Gini index is an integral

- First, rank all households by income.
- Then compute the percentage of households whose income is at most a given percentage of the country's total income.

# The Gini index is an integral

If the bottom *a* % of households earn *b* % of the total income, then plot the point (*a*,*b*) on the Lorenz curve.



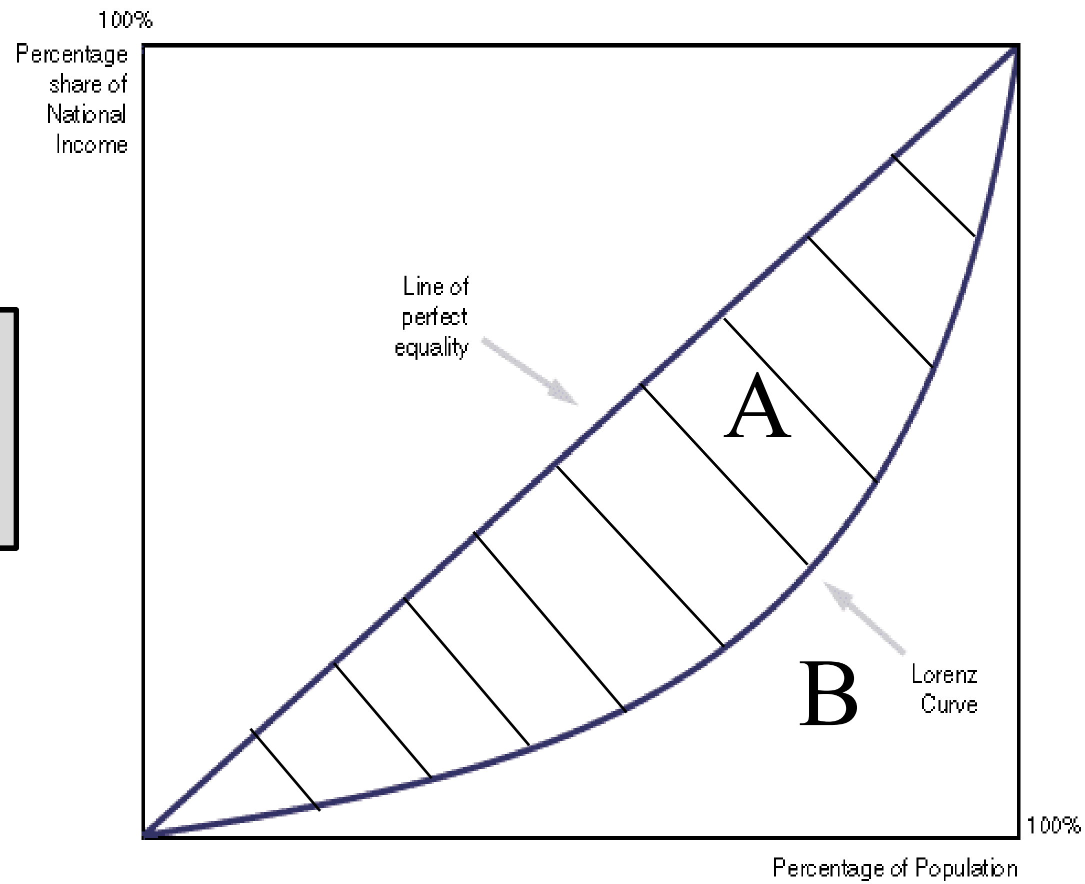
# The Gini index is an integral

**Definition:** The Gini index ***G*** is $A \,/\, (A + B)$

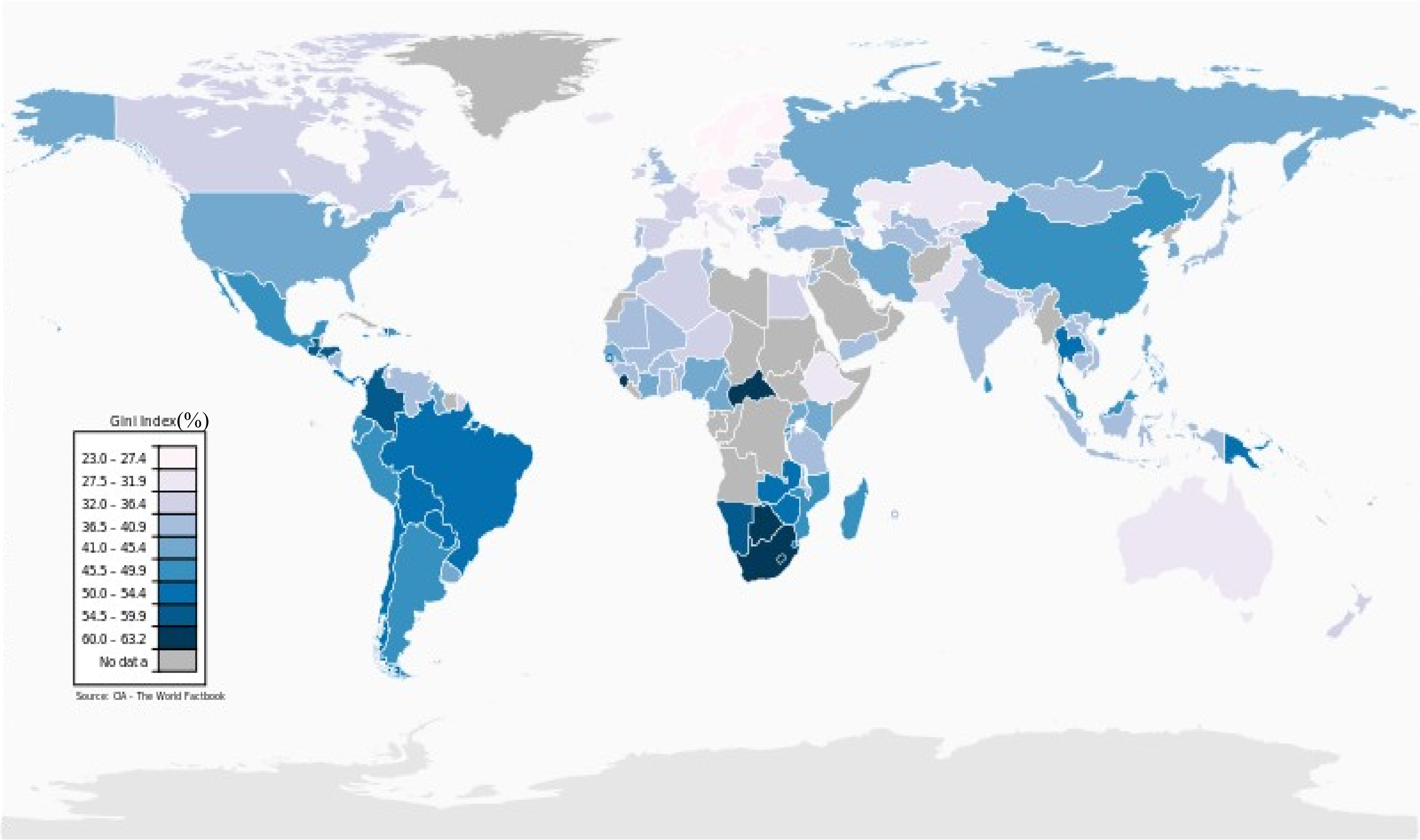
Now: $A + B = ½$

So: $G = 2\,A$

$$G = 2\int_0^1 (x - L(x))dx$$



# The Gini index is an integral




Netherlands: $G = 0.29$

# Integrals

- Theme: Area
- Theme: Riemann Sum
- Theme: Fundamental Theorem
- Theme: Antiderivatives

# Summarizing Exercise

Determine *dy/dx* in case *y* is given by

$$y(x) = x \int_1^x \frac{t}{1+t^4}\, dt$$

# Calculus 1 B

## -Contents-

- ❑√ Integrals
- ❑ Calculation techniques for integrals
- ❑ Power and Taylor series

- ❑ First order ODEs
- ❑ Complex numbers
- ❑ Second order ODEs

# See you next week!

Substitution …