Tag : Toetsen/18-19/ItM+Calc1A.18-19[02].CorrectionModel.EN
Course : **Introduction to Mathematics + Calculus 1A**
Date : Friday November 9th, 2018
Time : 13:45 – 16:45

Solutions

5. (a) [1 pt] With $\mathbf{u} = \overrightarrow{PQ} = \langle 1, 1, 4\rangle$ and $\mathbf{v} = \overrightarrow{PR} = \langle 1, 4, 1\rangle$:

$$\mathbf{u}\times\mathbf{v} = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 4 & 1 \end{bmatrix} \begin{matrix} 1 & 1 \\ 1 & 4 \end{matrix} = \boxed{\langle -15, 3, 3\rangle}$$

(b) [2 pt] The normal equation of the plane V is

$$\mathbf{n} \bullet (\mathbf{x} - \mathbf{p}) = 0,$$

with $\mathbf{n}$ a normal vector and $\mathbf{p}$ a support vector of V.
Using $\mathbf{n} = \mathbf{u}\times\mathbf{v}$ as a normal vector and for example $\mathbf{p} = \overrightarrow{OP}$ as support vector we obtain the equation

$$\begin{aligned} \langle -15, 3, 3\rangle \bullet \langle x-1, y+1, z\rangle &= 0, \\ -15x + 3y + 3z &= -18 \end{aligned}$$

which can be simplified to

$$\boxed{5x - y - z = 6} \quad \text{or} \quad \boxed{z = 5x - y - 6}$$

(c) [2 pt] If θ is the angle at P, then

$$\mathbf{u} \bullet \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta$$

Calculate the dot product of $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{u} \bullet \mathbf{v} = 1\cdot 1 + 1\cdot 4 + 4\cdot 1 = 9.$$

Calculate the lengths of $\mathbf{u}$ and $\mathbf{v}$:

$$|\mathbf{u}| = \sqrt{1^2+1^2+4^2} = 3\sqrt{2}. \quad \text{and} \quad |\mathbf{v}| = \sqrt{1^2+4^2+1^2} = 3\sqrt{2}$$

Therefore $\cos\theta = \dfrac{9}{3\sqrt{2}\cdot 3\sqrt{2}} = \frac{1}{2}$,

and consequently the angle at P is $\theta = \boxed{\frac{1}{3}\pi}$

(d) [2 pt] The projection of $\mathbf{u}$ onto $\mathbf{v}$ is

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}$$

With $\mathbf{u} \cdot \mathbf{v} = 9$ and $|\mathbf{v}|^2 = 18$ (see (c)), we obtain

$$\begin{aligned}\operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \frac{9}{18}\mathbf{v} \\ &= \tfrac{1}{2}\langle 1, 4, 1\rangle = \boxed{\left\langle \tfrac{1}{2}, 2, \tfrac{1}{2}\right\rangle}\end{aligned}$$

6. [2 pt] Step 1:

Note that the limit is of type "$\frac{0}{0}$", so the use of L'Hôpitals rule is justified.

Apply L'Hôpital' rule:

$$\lim_{x\to 0} \frac{x\, e^x}{e^{2x} - 1} = \lim_{x\to 0} \frac{e^x + x\, e^x}{2e^{2x}}. \qquad (*)$$

Step 2:

Use the quotient law for limits:

$$\lim_{x\to 0} \frac{e^x + x\, e^x}{2e^{2x}} = \lim_{x\to 0} \frac{e^0 + 0 \cdot e^0}{2e^{2\cdot 0}} = \boxed{\tfrac{1}{2}}$$

Note 1

Of course you can simplify the expression on the right-hand side of $(*)$, before taking the limit:

$$\lim_{x\to 0} \frac{e^x + x\, e^x}{2e^{2x}} = \lim_{x\to 0} \frac{1 + x}{2e^x} = \lim_{x\to 0} \frac{1 + 0}{2e^0} = \tfrac{1}{2}.$$

Note 2

Applying L'Hôpital on the right limit of $(*)$ is, although the results is correct, not justified, since the limit is *not* of type "$\frac{0}{0}$".

7. (a) [2 pt] Since $-1 \le \sin x \le 1$ for all x, we have

$$-|x| \le x\, \sin(\ln|x|) \le |x| \quad \text{for all } x \neq 0.$$

Now both $\lim_{x\to 0} |x| = 0$ and $\lim_{x\to 0} -|x| = 0$, so the sandwich theorem implies that $\lim_{x\to 0} f(x) = 0$.

(b) [1 pt] The function f is not continuous at 0, since $\lim_{x\to 0} f(x) \neq f(0)$.

8. (a) [2 pt] Using the definition of the derivative:

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{3h^2-h^3}-0}{h} \\ &= \lim_{h\to 0} \frac{|h|}{h}\sqrt{3-h}. \qquad (*) \end{aligned}$$

This limit depends on the sign of h:

$$\lim_{h\to 0^+} \frac{|h|}{h}\sqrt{3-h} = +\sqrt{3} \quad \text{and} \quad \lim_{h\to 0^-} \frac{|h|}{h}\sqrt{3-h} = -\sqrt{3},$$

so the limit $(*)$ does not exist.

(b) [2 pt] The derivative of $f(x)$ is

$$f'(x) = \frac{6x-3x^2}{2\sqrt{3x^2-x^3}} = \tfrac{3}{2} \cdot \frac{x}{|x|} \cdot \frac{2-x}{\sqrt{3-x}}.$$

Solving the equation $f'(x)=0$ gives $x=2$, so 2 is a critical point.

Also, from (a) follows that 0 is a critical point.

NOTE concluding $f'(0)=0$ because the numerator $6x-3x^2=0$ whenever $x=0$ is wrong.

(c) [1 pt] Candidates for the extreme values of f on $[1,3]$ are the boundaries 1 and 3, as well as the critical points in the interval $(1,3)$.

x	$f(x)$
1	$\sqrt{2}$
2	2
3	0

The absolute minimum is 0, and the absolute maximum is 2.

9. [2 pt] Use polar coordinates: $x = r\cos\theta$ and $x = r\sin\theta$. Then $f(x,y)$ becomes

$$f(r\cos\theta, r\sin\theta) = \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r} = r\big(\cos^2\theta - \sin^2\theta\big).$$

Since $\cos^2\theta - \sin^2\theta$ is a bounded function (for example: $|\cos^2\theta - \sin^2\theta| = |\cos(2\theta)| \le 1$, we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0^+} f(r\cos\theta, r\sin\theta) = 0.$$

An alternative argument is to use the sandwich theorem:

$$0 \leq |f(x,y)| = r\big(\cos^2\theta - \sin^2\theta\big) = r|\cos(2\theta)| \leq r \ \rightarrow \ 0$$

for $(x,y) \rightarrow 0$.

10. [3 pt] The equation for the tangent plane through $\big(a, b, f(a,b)\big)$ is

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)\big(x-a\big) + \frac{\partial f}{\partial y}(a,b)\big(y-b\big). \qquad (*)$$

Calculate the partial derivatives of f:

$$\frac{\partial f}{\partial x}(x,y) = \frac{2x}{x^2+y^2-1},$$
$$\frac{\partial f}{\partial y}(x,y) = \frac{2y}{x^2+y^2-1}.$$

Evaluate f and the partial derivatives at $(a,b) = (1,-1)$:

$$f\big(1,-1\big) = 0,$$
$$\frac{\partial f}{\partial x}(1,-1) = 2,$$
$$\frac{\partial f}{\partial y}(1,-1) = -2.$$

Write down the equation of the tangent plane (fill out all results in $(*)$):

$$z = 0 + 2(x-1) - 2\big(y-(-1)\big),$$
$$z = 2x - 2y - 4.$$

The equation may be rearranged, like

$$z + 4 = 2x - 2y,$$

or: $2x - 2y - z = 4.$

Total: 22 points