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Tag : Toetsen/18-19/Calc1A.18-19[01].CorrectionModel.EN
Course : **Calculus 1A**
Date : Friday October 26th, 2018
Time : 13:45 – 15:45

Solutions

1. (a) [1 pt] Define $\mathbf{u} = \overrightarrow{PQ} = \langle 1, 1, 4\rangle$ and $\mathbf{v} = \overrightarrow{PR} = \langle -1, 2, 2\rangle$, then

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} 1 & 1 & 4 \\ -1 & 2 & 2 \end{bmatrix} \begin{matrix} 1 & 1 \\ -1 & 2 \end{matrix} = \boxed{\langle -6, -6, 3\rangle}$$

(b) [1 pt] The area of the parallelogram $PQRS$ is equal to $|\mathbf{u} \times \mathbf{v}|$.
The length of $\mathbf{u} \times \mathbf{v}$ is $\sqrt{(-6)^2 + (-6)^2 + 3^2} = \boxed{9}$

(c) [2 pt] If θ is the angle at P, then

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$$

Calculate the dot product of $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{u} \cdot \mathbf{v} = 1 \cdot (-1) + 1 \cdot 2 + 4 \cdot 2 = 9.$$

Calculate the lengths of $\mathbf{u}$ and $\mathbf{v}$:

$$|\mathbf{u}| = \sqrt{1^2 + 1^2 + 4^2} = 3\sqrt{2} \quad \text{and} \quad |\mathbf{v}| = \sqrt{(-1)^2 + 2^2 + 2^2} = 3.$$

Therefore $\cos\theta = \dfrac{9}{3 \cdot 3\sqrt{2}} = \frac{1}{2}\sqrt{2}$,

and consequently the angle at P is $\theta = \boxed{\frac{1}{4}\pi}$

! **Note**

Unfortunately there are students that infer a specific order on the vertices (where the diagonals are PR and QS), and calculate the angle $\angle QPS$:

Calculate $\mathbf{w} = \mathbf{v} - \mathbf{u} = \langle -2, 1, -2\rangle$.
If θ is the angle at P, then

$$\mathbf{u} \cdot \mathbf{w} = |\mathbf{u}|\,|\mathbf{w}| \cos\theta$$

Calculate the dot product of $\mathbf{u}$ and $\mathbf{w}$:

$$\mathbf{u} \cdot \mathbf{w} = 1 \cdot (-2) + 1 \cdot 1 + 4 \cdot (-2) = -9.$$

Calculate the lengths of $\mathbf{u}$ and $\mathbf{w}$:

$|\mathbf{u}| = \sqrt{1^2+1^2+4^2} = 3\sqrt{2}$ and $|\mathbf{w}| = \sqrt{(-2)^2+1^2+(-2)^2} = 3.$

Therefore $\cos\theta = \dfrac{-9}{3\cdot 3\sqrt{2}} = -\frac{1}{2}\sqrt{2}$,

and consequently the angle at P is $\theta = \boxed{\frac{3}{4}\pi}$

(d) [1 pt] The normal equation of the plane V is

$$\mathbf{n} \bullet (\mathbf{x} - \mathbf{p}) = 0,$$

with $\mathbf{n}$ a normal vector and $\mathbf{p}$ a support vector of V.
Using $\mathbf{n} = \mathbf{u}\times\mathbf{v}$ as a normal vector and for example $\mathbf{p} = \overrightarrow{OP}$ as support vector we obtain the equation

$$\begin{aligned}\langle -6,-6,3\rangle \bullet \langle x+1, y+3, z\rangle &= 0,\\ -6x-6y+3z &= 24\end{aligned}$$

which can be simplified to

$$\boxed{2x+2y-z=-8} \quad \text{or} \quad \boxed{z = 2x+2y+8}$$

2. [2 pt] Note that the limit is of type "$\frac{0}{0}$", so the use of L'Hôpitals rule is justified.

Calculate the limit using L'Hôpital once:

$$\lim_{x\to 0} \frac{x\sin x}{1-\cos x} = \lim_{x\to 0} \frac{\sin x + x\cos x}{\sin x}.$$

There are several options to handle this limit.

Method 1: use L'Hôpital twice

Again the limit is of type "$\frac{0}{0}$".

Use L'Hôpitals rule once more.

$$\lim_{x\to 0} \frac{\sin x + x\cos x}{\sin x} = \lim_{x\to 0} \frac{2\cos x - x\sin x}{\cos x} = \frac{2\cdot 1 - 0\cdot 0}{1} = \boxed{2}$$

Method 2: use a standard limit

Use the standard limit $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$

$$\lim_{x\to 0} \frac{\sin x + x\cos x}{\sin x} = \lim_{x\to 0} 1 + \frac{\cos x}{\frac{\sin x}{x}} = 1 + \frac{1}{1} = \boxed{2}$$

3. (a) [1 pt] From $\lim\limits_{x\to 0^-} \frac{1}{x} = -\infty$ follows directly $\lim\limits_{x\to 0^-} e^{\frac{1}{x}} = 0$, hence

$$\lim_{x\to 0^-} f(x) = \frac{0-1}{0+1} = -1.$$

(b) [2 pt] Several correct solutions exist, including L'Hôpital. Here's a direct method:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\left(e^{\frac{1}{x}} - 1\right)\cdot e^{-\frac{1}{x}}}{\left(e^{\frac{1}{x}} + 1\right)\cdot e^{-\frac{1}{x}}}$$
$$= \lim_{x\to 0^+} \frac{1 - e^{-\frac{1}{x}}}{1 + e^{-\frac{1}{x}}} = \frac{1-0}{1+0} = 1.$$

(c) [1 pt] Argumentation 1:

From (a) and (b) follows that left and right limit of $f(x)$ for $x \to 0$ are different, so $\lim\limits_{x\to 0} f(x)$ does not exist. Therefore f is not continuous at 0.

If correctly explained:

Argumentation 2:

Suppose f is continuous at 0. Then $\lim\limits_{x\to 0} f(x) = f(0) = 0$, and consequently $\lim\limits_{x\to 0^-} f(x) = 0$. But this contradicts (a), therefore f is not continuous at 0.

4. (a) [1 pt] Using the definition of the derivative:

$$\frac{d\sqrt[3]{x}}{dx}(0) = \lim_{h\to 0} \frac{\sqrt[3]{0+h} - \sqrt[3]{0}}{h}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt[3]{h^2}} = \infty.$$

(b) [2 pt] The derivative of $f(x) = 3\sqrt[3]{x} - 4x = 3x^{1/3} - 4x$ is

$$f'(x) = x^{-\frac{2}{3}} - 4. = \frac{1}{\sqrt[3]{x^2}} - 4.$$

Solving the equation $f'(x) = 0$ gives $x^2 = \frac{1}{64}$, so $\frac{1}{8}$ and $-\frac{1}{8}$ are critical points.

Also, from (a) follows that 0 is a critical point.

(c) [2 pt] Candidates for the extreme values of f on $[-1, 8]$ are the boundaries

-1 and 8, as well as the critical points in the interval $(-1, 8)$.

x	$f(x)$
-1	1
$-\frac{1}{8}$	-1
0	0
$\frac{1}{8}$	1
8	-26

The absolute minimum is -26, and the absolute maximum is 1.

5. [3 pt] If polar coordinates are used, the calculation should look like this:

$$\lim_{(x,y)\to(0,0)} \frac{x+y^2}{\sqrt{x^2+y^2}} = \lim_{r\to 0^+} \frac{r\cos\theta + r^2\sin^2\theta}{r} = \lim_{r\to 0^+} \cos\theta + r\sin^2\theta = \cos\theta,$$

which depends on θ. So for example if $\theta = 0$ (approach $(0,0)$ along the positive x-axis), then the limit is 1. But if for example $\theta = \frac{1}{2}\pi$ (approach $(0,0)$ along the positive y-axis), then the limit is 0.

If approaching $(0,0)$ along a line is used, then the argumentation should look like this:

Let $y = \alpha x$ for some value of α, then

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along line } y=\alpha x}} \frac{x+y^2}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{1+\alpha^2 x}{\sqrt{1+\alpha^2}} = \frac{1}{\sqrt{1+\alpha^2}}.$$

So for instance, if (x,y) is on the x-axis, then $y = 0$, which can be achieved by choosing $\alpha = 0$. In that case the limit is 1.

If (x,y) is on the line $y = x$, then $\alpha = 1$, and consequently the limit is $\frac{1}{2}\sqrt{2}$.

6. (a) [2 pt] The equation for the tangent plane through $\big(a, b, f(a,b)\big)$ is

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)\big(x-a\big) + \frac{\partial f}{\partial y}(a,b)\big(y-b\big). \qquad (*)$$

Calculate the partial derivatives of f:

$$\frac{\partial f}{\partial x}(x,y) = 4x^3 - 1,$$
$$\frac{\partial f}{\partial y}(x,y) = 3y^2.$$

Evaluate f and the partial derivatives at $(a,b) = (1,-1)$:

$$f\big(1,-1\big) = 3,$$
$$\frac{\partial f}{\partial x}(1,-1) = 3,$$
$$\frac{\partial f}{\partial y}(1,-1) = 3.$$

Write down the equation of the tangent plane (fill out all results in $(*)$):

$$z = 3 + 3(x-1) + 3\big(y-(-1)\big),$$
$$z = 3 + 3x + 3y.$$

The equation may be rearranged, like

$$z = 3(x+y+1),$$
$$\text{or:} \quad x + y - \tfrac{1}{3}z = -1.$$

(b) [1 pt] From (a) follows: the linearization of f at $(-1,1)$ is

$$L(x,y) = 3x + 3y + 3.$$

An approximation of $f\big(\frac{4}{3}, -\frac{2}{3}\big)$ then is

$$L\big(\tfrac{4}{3}, -\tfrac{2}{3}\big) = 3 \cdot \tfrac{4}{3} - 3 \cdot \tfrac{2}{3} + 3 = 5.$$

Total: 22 points