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Course : **Mathematics B1**

Date : Friday October 27th, 2017

Time : 13:45 – 15:45

Solutions

1. Define $P(-1,1,1)$, $Q(1,3,0)$, and $R(-2,1,2)$.

(a) [1 pt] $\overrightarrow{PQ} = \langle 2,2,-1\rangle$ and $\overrightarrow{PR} = \langle -1,0,1\rangle$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} 2 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{matrix} 2 & 2 \\ -1 & 0 \end{matrix} = \boxed{\langle 2,-1,2\rangle}$$

(b) [2 pt] Use the normal equation $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$, with $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$ (see (a)) and $\mathbf{p} = \overrightarrow{OP}$.

$$\begin{aligned} &\langle 2,-1,2\rangle \cdot \big\langle x-(-1), y-1, z-1\big\rangle = 0 \\ &2(x+1) - (y-1) + 2(z-1) = 0 \\ &\boxed{2x - y + 2z = -1} \end{aligned}$$

(c) [2 pt] Let $\mathbf{u} = \overrightarrow{PQ} = \langle 2,2,-1\rangle$ and $\mathbf{v} = \overrightarrow{PR} = \langle -1,0,1\rangle$. Then the cosine of the angle θ at P is

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}.$$

Calculate the lengths and the dot product of $\mathbf{u}$ and $\mathbf{v}$:

$$\begin{aligned} |\mathbf{u}| &= \sqrt{2^2 + 2^2 + (-1)^2} = 3, \\ |\mathbf{v}| &= \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}, \\ \mathbf{u}\cdot\mathbf{v} &= -3. \end{aligned}$$

Therefore $\cos\theta = -\dfrac{1}{\sqrt{2}} = -\tfrac{1}{2}\sqrt{2}$,

and consequently $\theta = \dfrac{3\pi}{4}$.

Note: The answer $\theta = -3\pi/4$ is also correct. However, the answer $\theta = \pi/4$ and $\theta = -\pi/4$ are wrong.

2. [3 pt] Method 1: simplify the fraction:

$$\begin{aligned}\frac{x-1}{x-\sqrt{x}} &= \frac{(x-1)\big(x+\sqrt{x}\big)}{\big(x-\sqrt{x}\big)\big(x+\sqrt{x}\big)}\\ &= \frac{(x-1)\big(x+\sqrt{x}\big)}{x^2-x}\\ &= \frac{\cancel{(x-1)}\big(x+\sqrt{x}\big)}{x\cancel{(x-1)}}\\ &= \frac{x+\sqrt{x}}{x}\\ &= 1+\frac{1}{\sqrt{x}}\end{aligned}$$

hence

$$\lim_{x\to 1}\frac{x-1}{x-\sqrt{x}} = \lim_{x\to 1} 1+\frac{1}{\sqrt{x}} = 2.$$

Method 2: with l'Hôpital:

Observing that this limit is of type "$0/0$" (a calculation is not required).

$$\begin{aligned}\lim_{x\to 1}\frac{x-1}{x-\sqrt{x}} &= \lim_{x\to 1}\frac{1}{1-\dfrac{1}{2\sqrt{x}}}\\ &= \frac{1}{1-\frac{1}{2}} = 2.\end{aligned}$$

Method 3: Substitute $\sqrt{x}=u$:

Observe that $x=u^2$, and that $u\to 1$ whenever $x\to 1$.

$$\begin{aligned}\lim_{x\to 1}\frac{x-1}{x-\sqrt{x}} &= \lim_{u\to 1}\frac{u^2-1}{u^2-u}\\ &= \lim_{u\to 1}\frac{(u+1)\cancel{(u-1)}}{u\cancel{(u-1)}}\\ &= \lim_{u\to 1}\frac{u+1}{u}\\ &= \frac{1+1}{1} = 2.\end{aligned}$$

3. (a) [2 pt]

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \tan^{-1}\left(\frac{1}{x}\right)$$
$$= \lim_{u\to -\infty} \tan^{-1}(u))$$
$$= -\frac{\pi}{2}$$
$$= f(0)$$

(b) [2 pt] The function f is *not* continuous at 0.

With a similar calculation, show that $\lim_{x\to 0^+} f(x) = \pi/2$:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \tan^{-1}\left(\frac{1}{x}\right)$$
$$= \lim_{u\to +\infty} \tan^{-1}(u)) = +\frac{\pi}{2},$$

so $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$, and consequently $\lim_{x\to 0} f(x)$ does not exist.

Alternatively, you could argue: $\lim_{x\to 0^+} f(x) \neq f(0)$.

4. [4 pt] *Critical points on the interior of* $(0, 9)$*:*

Find the derivative of f:

$$f'(x) = 1 - \frac{1}{\sqrt{x}}$$

Solve $f'(x) = 0$:

$$f'(x) = 0 \quad \Leftrightarrow \quad 1 - \frac{1}{\sqrt{x}} = 0 \quad \Leftrightarrow \quad \sqrt{x} = 1 \quad \Leftrightarrow \quad x = 1,$$

so the only critical point of f is 1, which is element of $(0, 9)$.

Function values on the boundaries of $[0, 9]$*:*

$$f(0) = 0 - 2\sqrt{0} = 0$$
$$f(9) = 9 - 2\sqrt{9} = 3$$

Absolute extremes of f *on* $[0, 9]$*:*

x	$f(x)$	
0	0	
1	-1	$\leftarrow$ min
9	3	$\leftarrow$ max

So the absolute minimum is -1, and the absolute maximum is 3.

5. [3 pt] Use polar coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

(∗)

Convert $f(x,y)$ into a function of r and θ, using the identity $r^2 = x^2 + y^2$:

$$\begin{aligned} f(x,y) &= \frac{x\sqrt{|y|}}{\sqrt{x^2+y^2}} \\ &= \frac{\cancel{r}\cos\theta\sqrt{r|\sin\theta|}}{\cancel{r}} \\ &= \sqrt{r}\cos\theta\sqrt{|\sin\theta|}. \end{aligned} \qquad (*)$$

The expression $\cos\theta\,\sqrt{|\sin\theta|}$ is bounded. (∗∗)

Therefore

$$\lim_{(x,y)\to(0,0)} \frac{x\sqrt{|y|}}{\sqrt[3]{x^2+y^2}} = \lim_{r\to 0^+} \sqrt{r}\cos\theta\sqrt{|\sin\theta|} = 0.$$

(∗∗∗)

Don't forget the absolute value bars: $\sqrt{\sin\theta}$ is not defined whenever $\sin\theta < 0$.

6. [3 pt] The equation for the tangent plane through $\big(a, b, f(a,b)\big)$ is

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)\big(x-a\big) + \frac{\partial f}{\partial y}(a,b)\big(y-b\big). \qquad (*)$$

Calculate the partial derivatives of f:

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= 3x^2 + 2xy, \\ \frac{\partial f}{\partial y}(x,y) &= x^2 - 2y. \end{aligned}$$

Evaluate f and the partial derivatives at $(a,b) = (1,-1)$:

$$\begin{aligned} f(a,b) &= 1^3 + 1^2(-1) - (-1)^2 = -1, \\ \frac{\partial f}{\partial x}(1,-1) &= 1, \\ \frac{\partial f}{\partial y}(1,-1) &= 3. \end{aligned}$$

Note: for $f(a,b)$ one can also take the third coordinate of the given point $(1,-1,-1)$.

Write down the equation of the tangent plane (fill out all results in $(*)$):

$$\begin{aligned} z &= -1 + 1(x-1) + 3\big(y-(-1)\big), \\ z &= x + 3y + 1. \end{aligned}$$

Filling out all results in $(*)$ is not sufficient to award the last half point. “Simplified as much as possible” means the equation should contain x, y and z only once, and there should be only one constant term, like in

$$z = x + 3y + 1,$$

$$\text{or} \quad x + 3y - z = -1,$$

$$\text{or} \quad z - x - 3y = 1,$$

$$\text{or} \quad x + 3y - z + 1 = 0.$$