UNIVERSITY OF TWENTE.

INTRODUCTION TO MATHEMATICS LECTURE 2

JASPER DE JONG

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TXT 1

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What does $\exists n \in \mathbb{Z}(m = 2n)$ mean?

A. m is even.

B. Even numbers exist.

C. All numbers are even.

D. There exist numbers that are twice as large as other numbers.

The question will open when you start your session and slideshow.

Time: 120s

Internet TXT

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The applicable explanations will be inserted after you have started a session.

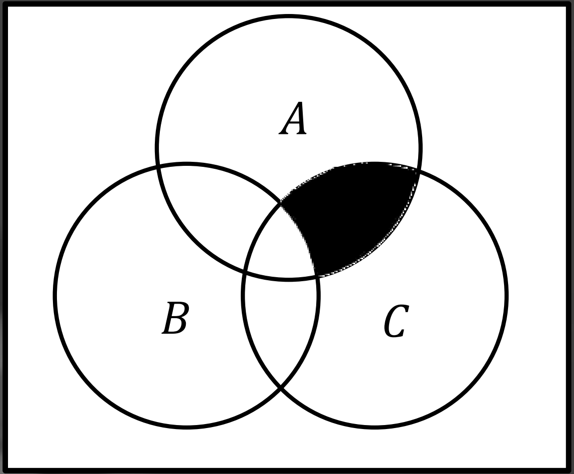
Votes: 1

What does ∃n∈ℤ(m=2n) mean?

A.	m is even.	100.0%
B.	Even numbers exist.	0.0%
C.	All numbers are even.	0.0%
D.	There exist numbers that are twice as large as other numbers.	0.0%

Closed

What does the black area represent?



A. A∪C

B. A∩C

C. (A∪C)−B

D. (A∩C)−B

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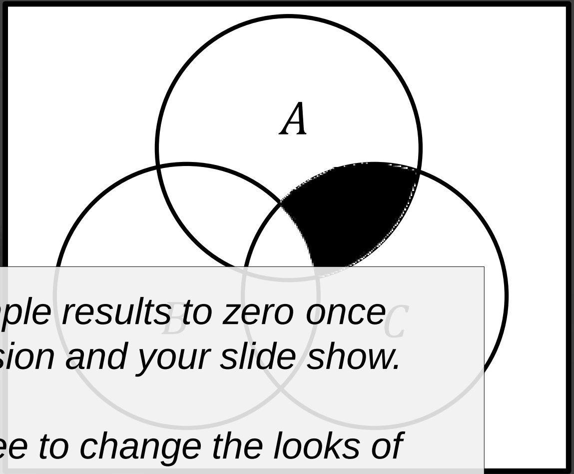
s: 1

What does the black area represent?



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A.	AUC	0.0%
B.	A∩C	0.0%
C.	(AUC)−B	0.0%
D.	(A∩C)−B	100.0%

Closed

TODAY'S LECTURE
PROOF TECHNIQUES

- Truth
  - Hard to find in real life.
  - Often easy to find in math…
    … when using the right techniques.

TODAY'S LECTURE
PROOF TECHNIQUES

- (Counter)example
- Direct proof
- Case distinction
- Proof by contradiction
- Mathematical Induction

TODAY'S LECTURE
PROOF TECHNIQUES

- **(Counter)example**
- **Direct proof**
- **Case distinction**
- Proof by contradiction
- Mathematical Induction

(COUNTER)EXAMPLE

Proposition

Cows don't exist

Counterexample:

Proposition

Every person has the same gender

Counterexample: 1 male and 1 female

Theorem

$\exists m \in \mathbb{Z}(\exists n \in \mathbb{Z}(m = 2n))$

Example: $m = 6, n = 3$

DIRECT PROOF

Theorem

There exist at least two Facebook users with the same number of Facebook friends.

Proof

- There are billions of users on Facebook. Denote the set of users by U.
- Each user has a number of friends from the set $F = \{0,1, \dots, 5000\}$.
- Since $|\mathrm{U}| > |F|$, at least two users have the same number of friends.

The last step uses the pigeonhole principle.

Theorem (Pigeonhole Principle)

When n items are put into m containers, where $n > m$, then at least one container contains more than one item.

CASE DISTINCTION

Theorem

Even without a limit to the number of friends, there exist at least two Facebook users with the same number of Facebook friends.

Proof

- Denote the number of users on facebook by n.
- Each user has a number of friends from the set $F = \{0,1,\ldots,n-1\}$.
  - **Case 1** Suppose no user has 0 friends. We have $|\{1,2,\ldots,n-1\}| = n-1 < n$. By the pigeonhole principle, the theorem holds.
  - **Case 2** Suppose some user u has 0 friends. Now no user has $n-1$ friends, since u is not her friend. Therefore $|\{0,1,\ldots,n-2\}| = n-1 < n$. By the pigeonhole principle, the theorem holds.

Abstraction allows us to use one theorem for different applications!

Consider the following statements:

1 If there are at least two students, there are always at least two students with the same number of students in their house.

2 If there are at least two students, there are always at least two students helping the same number of students with their study.

A. Both statements are false.

B. Statement 1 is false, and statement 2 is true.

C. Statement 1 is true, and statement 2 is false.

D. Both statements are true.

The question will open when you start your session and slideshow.

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Consider the following statements:

1 If there are at least two students, there are always at least two students with the same number of students in their house.

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A.	Both statements are false.	0.0%
B.	Statement 1 is false, and statement 2 is true.	0.0%
C.	Statement 1 is true, and statement 2 is false.	0.0%
D.	Both statements are true.	0.0%

Closed

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SOLUTION

Theorem

If there are at least two students, there are always at least two students with the same number of students in their house.

Proof

- Define two students to be 'Facebook friends' if they live in the same house.
- By the Theorem on Facebook friends, this theorem also holds.

Proposition

If there are at least two students, there are always at least two students helping the same number of students with their study.

Counterexample

Consider two students $\{1,2\}$. 1 helps 2, but 2 does not help 1.

CASE DISTINCTION

WHERE DOES HELPING WITH STUDY FAIL?

Theorem

Even without a limit to the number of friends, there exist at least two Facebook users with the same number of Facebook friends.

Proof

- Denote the number of users on facebook by n.
- Each user has a number of friends from the set $F = \{0,1,\dots,n-1\}$.
  - **Case 1** Suppose no user has 0 friends. $|\{1,2,\dots,n-1\}| = n-1 < n$. By the pigeonhole principle, the theorem holds.
  - **Case 2** Suppose some user u has 0 friends. Now no user has $n-1$ friends, since u is not her friend. Therefore $|\{0,1,\dots,n-2\}| = n-1 < n$. By the pigeonhole principle, the theorem holds.

TODAY'S LECTURE
PROOF TECHNIQUES

- (Counter)example
- Direct proof
- Case distinction
- **Proof by contradiction**
- Mathematical Induction

PROOF BY CONTRADICTION

Theorem

$\sqrt{2} \notin \mathbb{Q}$

Proof

- Suppose $\sqrt{2} \in \mathbb{Q}$, i.e. (that is) $\exists p, q \in \mathbb{Z}\left(\sqrt{2} = \frac{p}{q}\right)$
- Assume w.l.o.g. (without loss of generality) that p and q have no common divisor. (if they do, divide both by this divisor.)
- $\sqrt{2} = \frac{p}{q} \;\Rightarrow\; 2 = \frac{p^2}{q^2} \;\Rightarrow\; 2q^2 = p^2$
  $\Rightarrow$ p^2 is even $\Rightarrow$ p is even (Try to prove this yourself.)
  $\Rightarrow \exists n \in \mathbb{Z}(p = 2n)$
  $\Rightarrow 2q^2 = p^2 = (2n)^2 = 4n^2 \;\Rightarrow\; q^2 = 2n^2$
  $\Rightarrow$ q^2 is even $\Rightarrow$ q is even
  $\Rightarrow$ 2 divides both p and q $\Rightarrow$ **contradiction!**

Which of the following statements are true?

1 $\sqrt{\sqrt{2}} \in \mathbb{Q}$

2 $\sqrt{3} \in \mathbb{Q}$

A. Both statements are false.

B. Statement 1 is false and statement 2 is true.

C. Statement 1 is true and statement 2 is false.

D. Both statements are true.

The question will open when you start your session and slideshow.

Time: 120s

Votes: 1

Which of the following statements are true?

1 $\sqrt{\sqrt{2}} \in \mathbb{Q}$

2 $\sqrt{3} \in \mathbb{Q}$

A. Both statements are false. 0.0%

B. Statement 1 is false and statement 2 is true. 0.0%

C. Statement 1 is true and statement 2 is false. 0.0%

D. Both statements are true. 0.0%

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Closed

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PROOF BY CONTRADICTION

Theorem

$\sqrt{\sqrt{2}} \notin \mathbb{Q}$

Proof

- Suppose $\sqrt{\sqrt{2}} \in \mathbb{Q}$, i.e. $\exists p, q \in \mathbb{Z}\left(\sqrt{\sqrt{2}} = \frac{p}{q}\right)$
- $\sqrt{2} = \frac{p^2}{q^2} \;\Rightarrow\; \sqrt{2} \in \mathbb{Q} \;\Rightarrow\;$ contradiction

Theorem

$\sqrt{3} \notin \mathbb{Q}$

Proof similar to $\sqrt{2} \notin \mathbb{Q}$ (exercise)

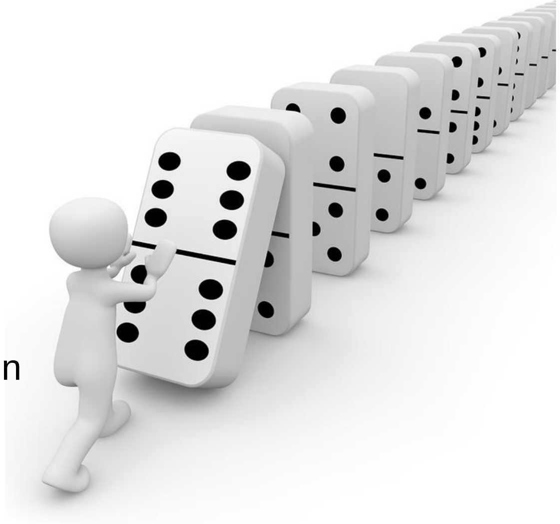
TODAY'S LECTURE
PROOF TECHNIQUES

- (Counter)example
- Direct proof
- Case distinction
- Proof by contradiction
- **Mathematical Induction**

MATHEMATICAL INDUCTION
INTUITION

- Given
  - The first domino falls.
  - If a domino falls, the next domino will also fall.
- Questions:
  - Does domino 2 fall?
  - Does domino 10 fall?
  - Does domino 7483946843209 fall?

- Mathematical induction lets us conclude from what is given that all domino’s will indeed fall.

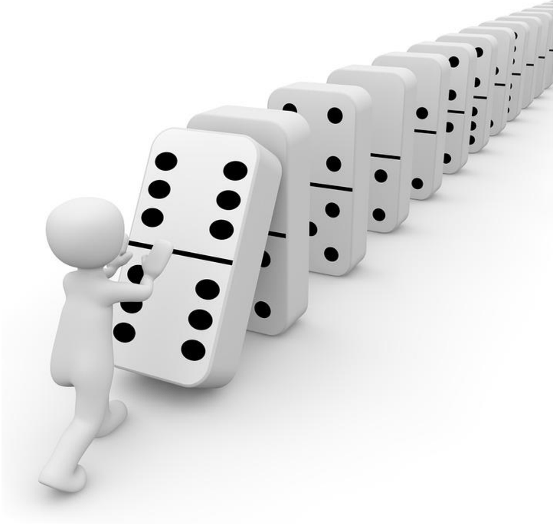


MATHEMATICAL INDUCTION
INTUITION

Given

- The first domino falls.
- If a domino falls, the next domino will also fall.

Then all domino's will fall!



Definition (Mathematical Induction)

Consider a predicate $S(n), n \in \mathbb{N}$.
Suppose that:
1. $S(1)$ (is true)
2. $\forall k \in \mathbb{N}\big(S(k) \rightarrow S(k+1)\big)$

Then $S(n)$ is true for all $n \in \mathbb{N}$

MATHEMATICAL INDUCTION

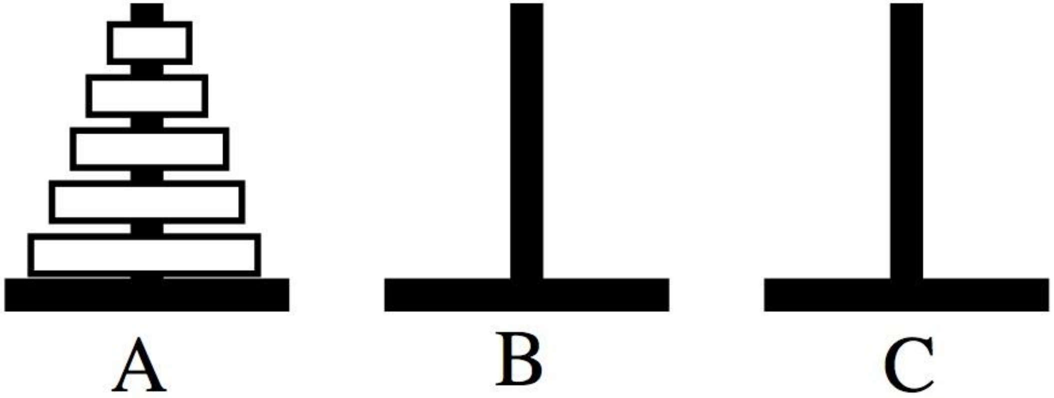
TOWERS OF HANOI

Goal

Move all disks from rod A to rod C

Rules

- Move one disk at a time
- No disk can be placed on top of a smaller disk



MATHEMATICAL INDUCTION

TOWERS OF HANOI

How many moves are needed to move a tower of n disks?

$$\text{Moves}(k+1) = \ \dots$$

MATHEMATICAL INDUCTION
TOWERS OF HANOI

How many moves are needed to move a tower of n disks?

$$\text{Moves}(k+1) = \text{Moves}(k) + \cdots$$

MATHEMATICAL INDUCTION
TOWERS OF HANOI

How many moves are needed to move a tower of n disks?

$$\text{Moves}(k+1) = \text{Moves}(k) + 1 + \cdots$$

MATHEMATICAL INDUCTION

TOWERS OF HANOI

How many moves are needed to move a tower of n disks?

$$\text{Moves}(k+1) = \text{Moves}(k) + 1 + \text{Moves}(k)$$

MATHEMATICAL INDUCTION

TOWERS OF HANOI

How many moves are needed to move a tower of n disks?

$$\text{Moves}(k+1) = 2 \cdot \text{Moves}(k) + 1$$

MATHEMATICAL INDUCTION
TOWERS OF HANOI

How many moves are needed to move a tower of n disks?

Lemma

$\forall k \in \mathbb{N}(\text{Moves}(k+1) = 2 \cdot \text{Moves}(k) + 1)$

Disks n	Moves(n)
1	1
2	3
3	7
4	15

Theorem

$\forall n \in \mathbb{N}(\text{Moves}(n) = 2^n - 1)$

MATHEMATICAL INDUCTION

EXAMPLE

Theorem

$\forall n \in \mathbb{N}(\text{Moves}(n) = 2^n - 1)$

$S(n)$: Moving a tower of n disks takes $2^n - 1$ moves.

Step 1 (basis step): Prove $S(1)$:

Mathematical Induction

Consider a predicate $S(n), n \in \mathbb{N}$.
Suppose that:

1. $S(1)$ (is true) ✓
2. $\forall k \in \mathbb{N}\big(S(k) \to S(k+1)\big)$

Then $S(n)$ is true for all $n \in \mathbb{N}$

Proof of $S(1)$

$\text{Moves}(1) = 2^1 - 1 = 1$, the number of moves required to move 1 disk.

MATHEMATICAL INDUCTION

EXAMPLE

Theorem

$\forall n \in \mathbb{N}(\text{Moves}(n) = 2^n - 1)$

$S(n)$: Moving a tower of n disks takes $2^n - 1$ moves.

Step 2 (induction step): Prove $\forall k \in \mathbb{N}\big(S(k) \rightarrow S(k+1)\big)$.

Mathematical Induction

Consider a predicate $S(n), n \in \mathbb{N}$.
Suppose that:

1. $S(1)$ (is true) ✓
2. $\forall k \in \mathbb{N}\big(S(k) \rightarrow S(k+1)\big)$ ✓

Then $S(n)$ is true for all $n \in \mathbb{N}$ ✓

Proof of $\forall k \in \mathbb{N}\big(S(k) \rightarrow S(k+1)\big)$

- Suppose for some k, $S(k)$: Moving k disks takes $2^k - 1$ moves. (Induction Hypothesis)
- We now need to show $S(k+1)$: Moving $k+1$ disks takes $2^{k+1} - 1$ moves.

$\text{Moves}(k+1) = 2 \cdot \text{Moves}(k) + 1$ (Lemma)

$= 2 \cdot \left(2^k - 1\right) + 1$ (By Induction Hypothesis)

$= 2^{k+1} - 2 + 1 = 2^{k+1} - 1$

Lemma

$\forall k \in \mathbb{N}(\text{Moves}(k+1) = 2 \cdot \text{Moves}(k) + 1)$

How much is 1+2+3+…+100?

A. 1050
B. 2500
C. 5050
D. 10100

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How much is $1 + 2 + 3 + \cdots + 100$?

A.	1050	0.0%
B.	2500	0.0%
C.	5050	0.0%
D.	10100	0.0%

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Closed

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STORY TIME

CARL FRIEDRICH GAUSS

Teacher: "What is $1 + 2 + 3 + \cdots + 100$?"

Gauss: "5050"

Theorem

$$\forall n \in \mathbb{N} \left(\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \right)$$

$$S(n)\colon 1 + 2 + 3 + \cdots + n \left(= \sum_{i=1}^{n} i \right) = \frac{n(n+1)}{2}$$

Step 1 (basis step): Prove $S(1)$.

Proof of $S(1)$

$$\sum_{i=1}^{1} i = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

Mathematical Induction

Consider a predicate $S(n), n \in \mathbb{N}$.
Suppose that:

1. $S(1)$ (is true)
2. $\forall k \in \mathbb{N}\big(S(k) \rightarrow S(k+1)\big)$

Then $S(n)$ is true for all $n \in \mathbb{N}$

STORY TIME

CARL FRIEDRICH GAUSS

Theorem

$$\forall n \in \mathbb{N}\left(\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\right)$$

$$S(n): 1 + 2 + 3 + \cdots + n \left(= \sum_{i=1}^{n} i\right) = \frac{n(n+1)}{2}$$

Mathematical Induction

Consider a predicate $S(n), n \in \mathbb{N}$.
Suppose that:

1. $S(1)$ (is true)
2. $\forall k \in \mathbb{N}\big(S(k) \rightarrow S(k+1)\big)$

Then $S(n)$ is true for all $n \in \mathbb{N}$

Proof of $\forall k \in \mathbb{N}\big(S(k) \rightarrow S(k+1)\big)$

- Suppose for some k, $S(k): \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ is true. (Induction Hypothesis)
- We now need to show that $S(k+1): \sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$ is true as well.
- $\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + (k+1)$

$$= \frac{k(k+1)}{2} + (k+1)$$ (Using the Induction Hypothesis)

$$= (k+1)\left(\frac{k}{2} + 1\right) = \frac{(k+1)(k+2)}{2}$$

MATHEMATICAL INDUCTION

EXAMPLE 3

Fake Theorem

Every person has the same gender

$S(n)$: For every group of size n, all persons in that group have the same gender.

Step 1 (basis step): Each person has the same gender as him/herself.

Step 2 (induction step): Prove $\forall k \in \mathbb{N}(S(k) \to S(k+1))$.

Mathematical Induction

Consider a predicate $S(n), n \in \mathbb{N}$.
Suppose that:

1. $S(1)$ (is true) ✓
2. $\forall k \in \mathbb{N}(S(k) \to S(k+1))$ ✓

Then $S(n)$ is true for all $n \in \mathbb{N}$ ✓

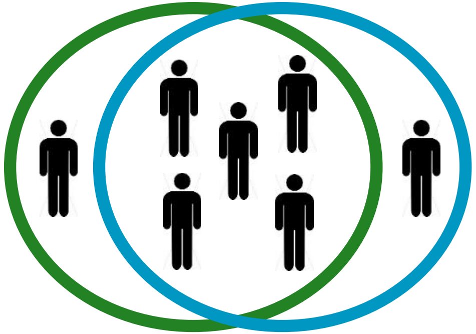
Proof of $\forall k \in \mathbb{N}(S(k) \to S(k+1))$

- Suppose for some k, $S(k)$: In every group of size k, each person has the same gender. (Induction Hypothesis)
- We now need to show that $S(k+1)$: In each group $\{1,2,\ldots,k+1\}$ of size $k+1$ each person has the same gender.
- By the induction hypothesis, both $\{1,2,\ldots,k\}$ and $\{2,3,\ldots,k+1\}$ have the same gender. This is the same gender for all $\{1,2,\ldots,k+1\}$, since it is shared by $\{2,3,\ldots,k\}$.

MATHEMATICAL INDUCTION
EXAMPLE 3

Fake Theorem

Every person has the same gender



$S(n)$: For every group of size n, all persons in that group have the same gender.

Step 1 (basis step): Each person has the same gender as him/herself.

Step 2 (induction step): Prove $\forall k \in \mathbb{N}\big(S(k) \rightarrow S(k+1)\big)$.

Proof of $\boldsymbol{\forall k \in \mathbb{N}\big(S(k) \rightarrow S(k+1)\big)}$

- Suppose for some k, $S(k)$: In every group of size k, each person has the same gender. (Induction Hypothesis)
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SUMMARY

- Proof Techniques
  - (Counter)example
  - Direct proof
  - Case distinction
  - Proof by contradiction
  - Mathematical Induction
- Abstraction helps us to use theorems for multiple applications!
- Be very critical about your proof!
- Sum notation: $\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2 = 14$