Calculus 1A

UNIVERSITY OF TWENTE.

Lecture 4

Functions of several variables

Overview

This lecture is based on the material of chapter 13 of Thomas' Calculus.

Topics:

- Functions of more than one variable.
- Graphs and level curves.
- Limits and continuity.
- Partial derivatives.
- The tangent plane.
- Linearization.

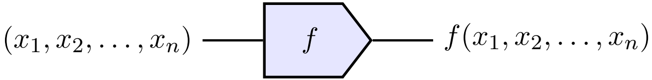
Functions of more than one variable

- The n-dimensional real space $\mathbb{R}^n$ is defined as the set

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \,|\, x_i \in \mathbb{R}\}.$$

Definition

Let D be a subset of $\mathbb{R}^n$. A **real valued function** f **on** D is a rule that assigns a unique real number $f(x_1, x_2, \ldots, x_n)$ to each point $(x_1, x_2, \ldots, x_n) \in D$.



The graph of a function on $\mathbb{R}^2$

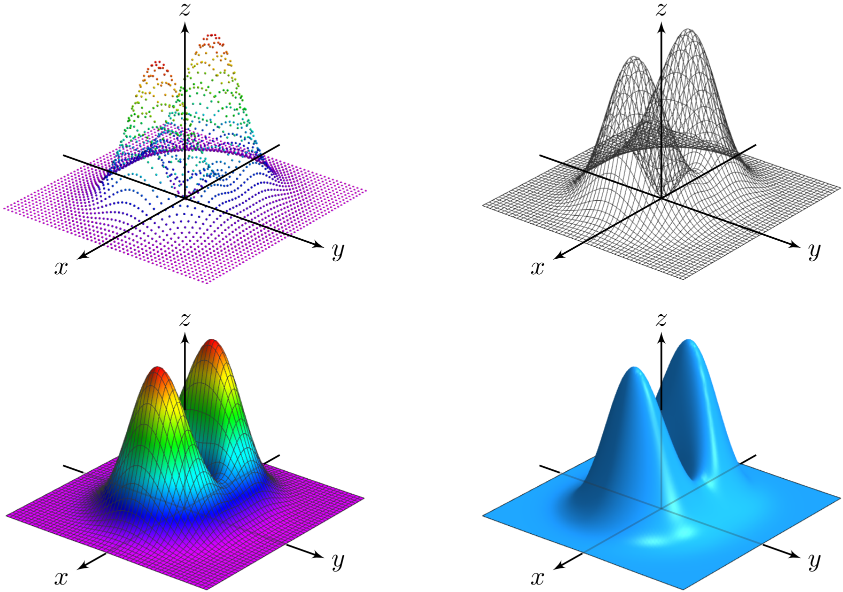
Definition

Let $f\colon D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a real valued functions. The **graph of f** is defined as the set

$$\left\{ \left(x_1, \ldots, x_n, f(x_1, \ldots, x_n)\right) \in \mathbb{R}^{n+1} \;\middle|\; (x_1, \ldots, x_n) \in D \right\}$$

- The graph of a function of n variables is a subset of the $(n+1)$-dimensional real space $\mathbb{R}^{n+1}$.
- The graph of a function of 2 variables is a surface in $\mathbb{R}^3$.

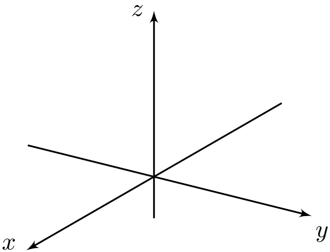
The graph of a function on $\mathbb{R}^2$

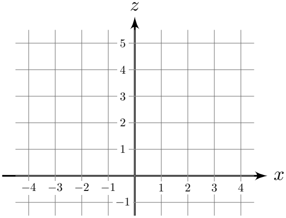


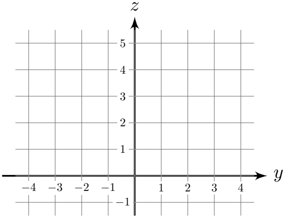
How humans draw the graph of a function on $\mathbb{R}^2$

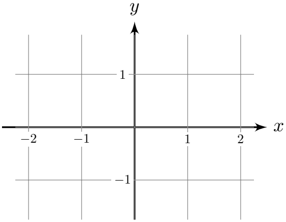
Problem

Sketch the graph of $z = 4 - x^2 - 4y^2$ above the xy-plane.

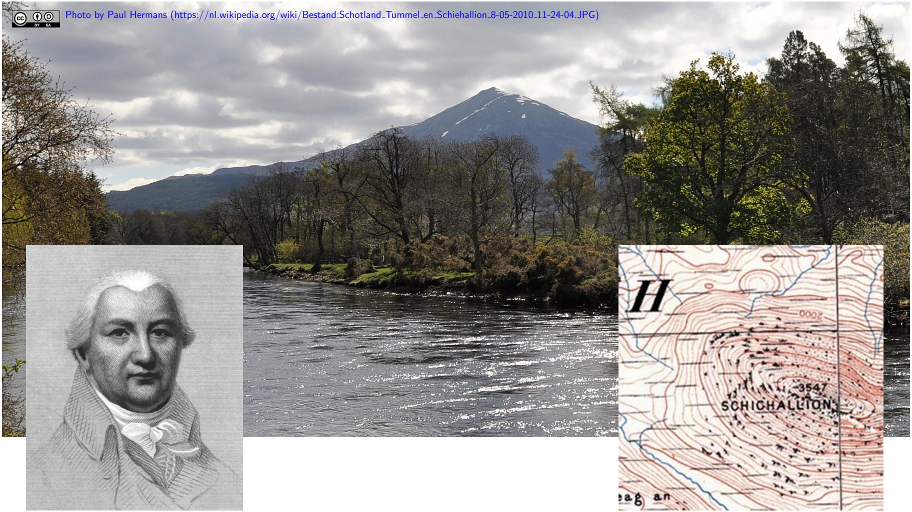








Level curves



- Charles Hutton (1774) was the first to use level curves.

Level curves

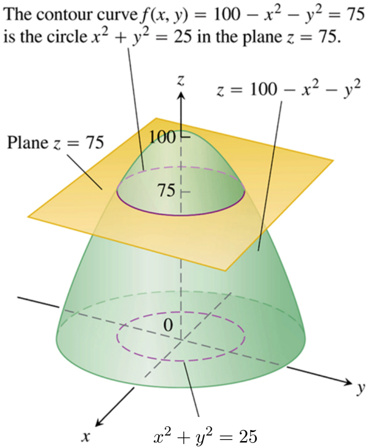
Definition

Let $f\colon D \to \mathbb{R}$ be a funcion with domain $D \subseteq \mathbb{R}^2$, and let c be a real number. The **level curve at level** c is the set

$$\left\{(x, y) \in \mathbb{R}^2 \;\middle|\; f(x, y) = c\right\}.$$

- The name is a misnomer: a level set doesn't need to be a curve.
- A level curve can even be the empty set.
- Alternative names are: **contour** and **contour line**.

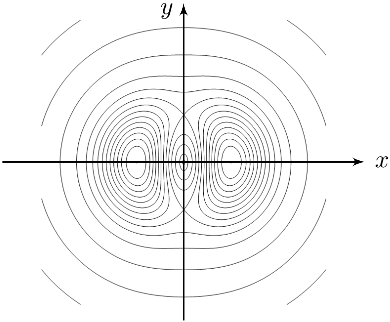
Level curves

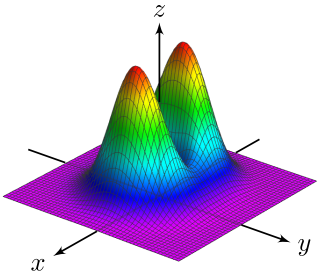


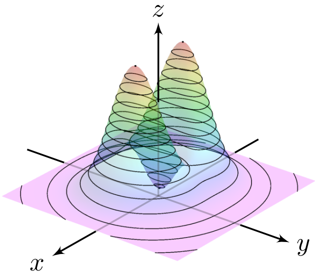
- Sometimes the curve $\{(x,y,c) \in \mathbb{R}^3 \mid f(x,y) = c\}$ is also called a contour.

Level curves

$$f(x,y) = (4x^2 + y^2)e^{-x^2-y^2}$$





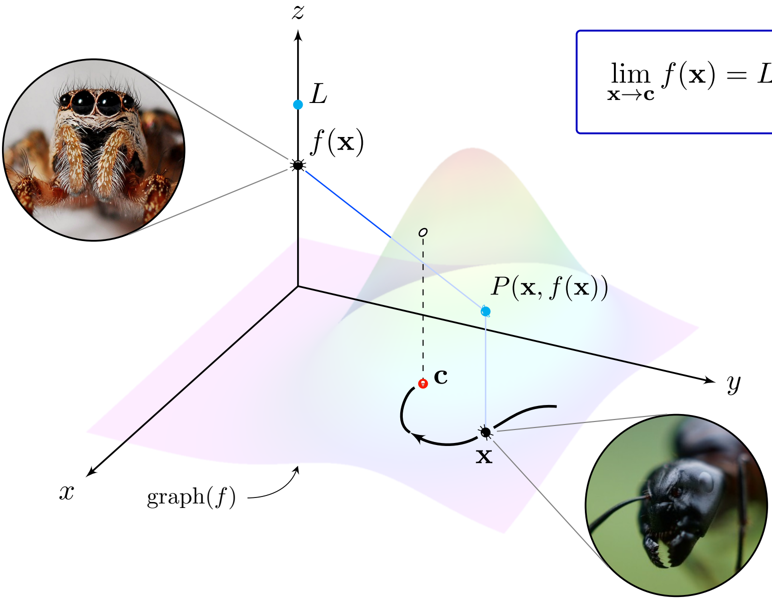


Limits and continuity

Australian peacock spider (Maratus Splendens), photo by Jurgen Otto

Limits of functions on $\mathbb{R}^2$: the spider-ant model

$$\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L$$



Limits of functions on $\mathbb{R}^2$

- Let f be a function of one variables. Let $c \in \mathbb{R}$, then $\lim_{x \to c} f(x) = L$ is defined as:

$$\forall\, \varepsilon > 0\; \exists\, \delta > 0\; \forall\, x \in \mathbb{R} \Big[0 < |x - c| < \delta \;\Rightarrow\; |f(x) - L| < \varepsilon \Big].$$

- Now let f be a function of two variables: Let $\mathbf{c} \in \mathbb{R}^2$, then $\lim_{\mathbf{x} \to \mathbf{c}} f(\mathbf{x}) = L$ is defined as:

$$\forall\, \varepsilon > 0\; \exists\, \delta > 0\; \forall\, \mathbf{x} \in \mathbb{R}^2 \Big[0 < |\mathbf{x} - \mathbf{c}| < \delta \;\Rightarrow\; |f(\mathbf{x}) - L| < \varepsilon \Big],$$

  where the leftmost pair of vertical bars "$|\ldots|$" indicate Euclidian length.

Limits of functions on $\mathbb{R}^2$

- Consequences of the similarity of the definitions for limits of functions on $\mathbb{R}$ and on $\mathbb{R}^2$:

  1. Limit laws, like the sum law, the product law, et cetera, work the same way.
  2. The replacement rules are similar (in stead of an interval surrounding c, use an open circular disc with center $\mathbf{c}$).
  3. There is a sandwich theorem.
  4. Definition and properties of continuity are automatically inferred.

- You can easily generalize to functions of *arbitrary* numbers of variables.

Definition

Let $f\colon D \subseteq \to \mathbb{R}^n$ be a function of n variables, and let $\mathbf{c} \in D$. The function f is **continuous at $\mathbf{c}$** if

$$\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = f(\mathbf{c}).$$

Practical approach:

1. $f(\mathbf{c})$ exists $\rightarrow$ $\mathbf{c}$ lies in the domain of f;

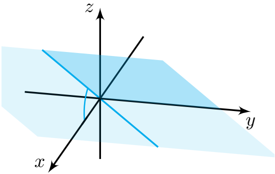
2. $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x})$ exists $\rightarrow$ f has a limit as $\mathbf{x}$ approaches $\mathbf{c}$;

3. $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = f(\mathbf{c})$ $\rightarrow$ the limit equals the function value.

Elementary limits

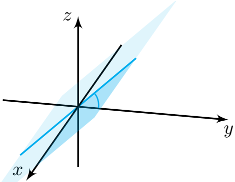
The function $(x, y) \mapsto x$ is continuous

$$\lim_{(x,y)\to(a,b)} x = a$$



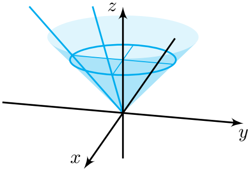
The function $(x, y) \mapsto y$ is continuous

$$\lim_{(x,y)\to(a,b)} y = b$$



The function $\mathbf{x} \mapsto |\mathbf{x}|$ is continuous

$$\lim_{\mathbf{x}\to\mathbf{c}} |\mathbf{x}| = |\mathbf{c}|$$



Calculating limits

Example

Calculate $\displaystyle\lim_{(x,y)\to(1,0)} \frac{\sin(\pi x)+e^y}{x^2+y^2}$.

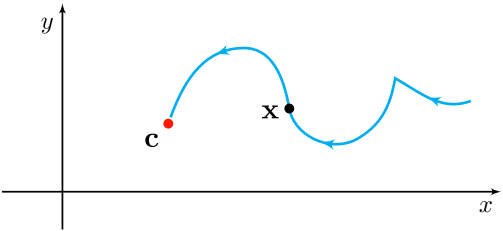
Path limits

- With regard to limits, there is an important difference with functions of one variable:

There are no one-sided limits.

Definition

A **path limit** is the limit of $f(\mathbf{x})$ where $\mathbf{x}$ approaches $\mathbf{c}$ along a continuous path ending in $\mathbf{c}$.



Path limits

Theorem

The following statements are equivalent:

(i) *The limit* $\lim_{\mathbf{x} \to \mathbf{c}} f(\mathbf{x})$ *exists.*

(ii) *All path limits of* $f(\mathbf{x})$ *along continuous paths ending in* $\mathbf{c}$ *exist, and have the same value.*

Warning

In order to conclude (i) from (ii) you must show that path limits exist and are the same along **every possible path**!

It is not sufficient to show the existence along straight lines only. See example 6 in section 14.2 of *Thomas' Calculus*.

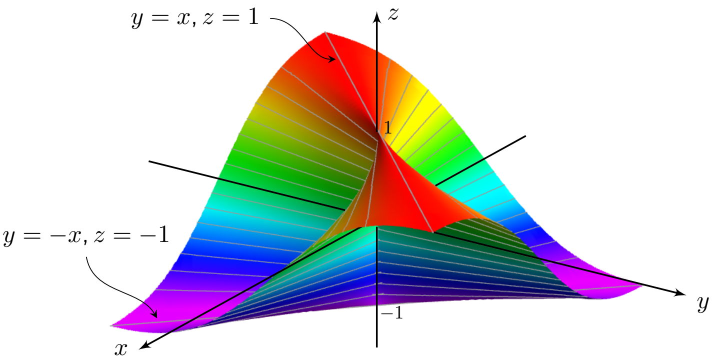
Showing that a limit does not exist

Theorem

1. *If a path limit along a continuous path ending at* $\mathbf{c}$ *does not exist, then* $\lim_{\mathbf{x} \to \mathbf{c}} f(\mathbf{x})$ *does not exist.*
2. *If two path limits along continuous paths ending at* $\mathbf{c}$ *do exist, but they are not the same, then* $\lim_{\mathbf{x} \to \mathbf{c}} f(\mathbf{x})$ *does not exist.*

Example

The limit of the function $f(x,y) = \dfrac{2xy}{x^2+y^2}$ for $\mathbf{x} \to \mathbf{0}$ does not exist.



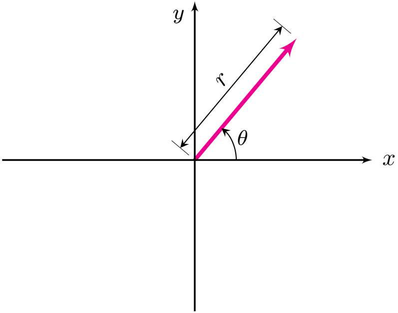
Example (continued)

- $\lim_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+y^2}$ does not exist.

Polar coordinates

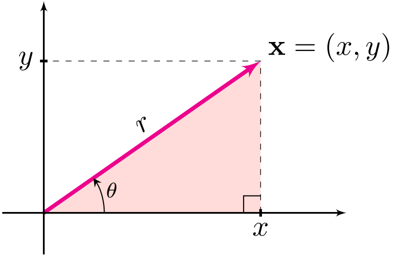
Photo by Mahufi

Polar coordinates



- Every point $(x, y) \neq (0, 0)$ is described by two **polar coordinates** r and θ.
- The number r is called the **radius**, and is defined as the distance to $(0, 0)$.
- The number θ is called the **polar angle**, and is defined as the angle that the vector $\mathbf{x}$ makes with the positive x-axis.

Coordinate transformations



- From polar coordinates to Cartesian coordinates:

$$x = r\cos\theta,$$
$$y = r\sin\theta.$$

- From Cartesian coordinates to polar coordinates:

$$r = \sqrt{x^2 + y^2},$$
$$\theta = \arctan(x, y).$$

Limits and polar coordinates

Theorem

Let f be a function of two variables. If

$$\lim_{r\to 0^+} f(r\cos\theta, r\sin\theta) = 0$$

independently from θ, then $\lim_{(x,y)\to(0,0)} f(x,y) = 0$*.*

Limits and polar coordinates

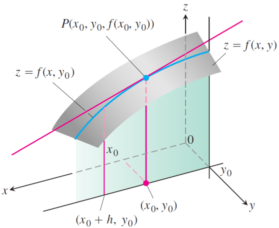
Example

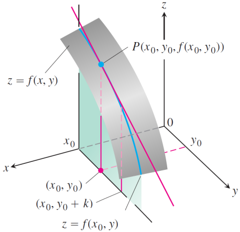
Show that $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0.$

Example

Try to calculate $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{y}$.

Partial derivatives





Definition

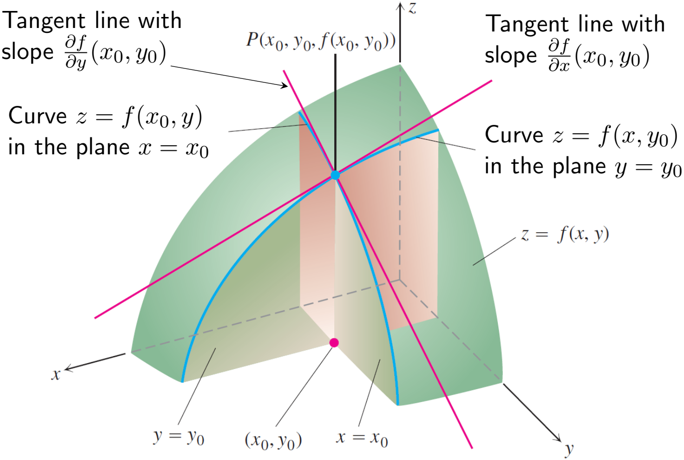
- The **partial derivative of f with respect to x** at (x_0, y_0) is

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}.$$

- The **partial derivative of f with respect to y** at (x_0, y_0) is

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

Partial derivatives



- The partial derivatives are the slopes of the tangent lines to the plane $x = x_0$ and $y = y_0$ respectively.

Notations

- The notation $\left.\dfrac{\partial f}{\partial x}\right|_{(x_0,y_0)}$ means
  1. first calculate $\dfrac{\partial f}{\partial x}(x,y)$;
  2. then replace x by x_0, and y by y_0 respectively.
- Alternative notations for partial derivatives:

$$\begin{array}{ccccc} \dfrac{\partial f}{\partial x} & \to & \partial_1 f & D_1 f & f_x \\ \dfrac{\partial f}{\partial y} & \to & \partial_2 f & D_2 f & f_y \end{array}$$

- Also, if $z = f(x,y)$, then

$$\begin{array}{ccc} \dfrac{\partial f}{\partial x} & \to & \dfrac{\partial z}{\partial x} \\ \dfrac{\partial f}{\partial x} & \to & \dfrac{\partial z}{\partial y} \end{array}$$

How to calculate partial derivatives

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

- The partial derivative $\dfrac{\partial f}{\partial x}(x_0, y_0)$ is equal to $g'(x_0)$ where

$$g(x) = f(x, y_0).$$

- According to the definition, the partial derivative with respect to x is the derivative of the function $f(x, y)$ where y remains constant.
- Similarly: $\dfrac{\partial f}{\partial y}(x, y)$ is the derivative of the function $f(x, y)$ where x remains constant.
- For partial derivatives, all methods, properties, and differentiation laws apply.

How to calculate partial derivatives

Example

Calculate the partial derivatives of $f(x, y) = x^2 + 3xy + y - 1$ at $(4, -5)$.

How to calculate partial derivatives

Example

Find $\dfrac{\partial f}{\partial y}$ of $f(x,y) = y\sin(xy)$, as a function of x and y.

Definition

The second-order partial derivatives van f are defined as follows:

$$f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(x,y)\right)$$

$$f_{yx}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right)$$

$$f_{xy}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right)$$

$$f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}(x,y)\right)$$

- Mnemonic: $f_{xy} = (f_x)_y$.

Mixed higher order partial derivatives

Theorem (Mixed Derivative Theorem)

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx}, exist and are continuous on an open environment of (a, b), then

$$\boxed{f_{xy}(a, b) = f_{yx}(a, b)}$$

- This theorem is also known a **Clairauts theorem**.
- **Example:**

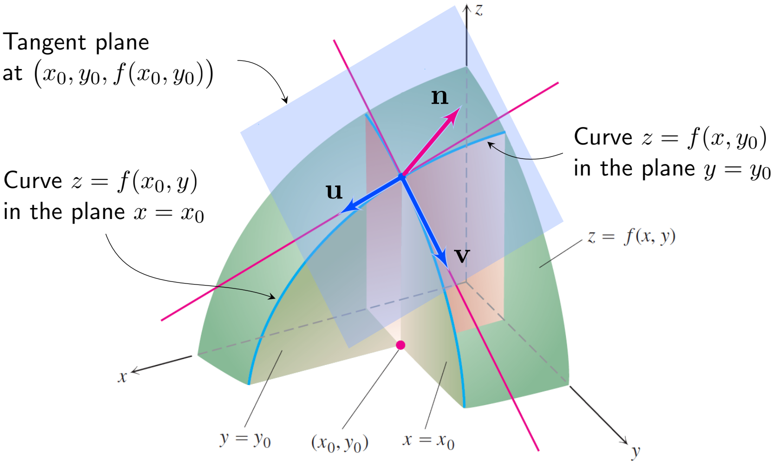
$$\begin{array}{ccccc} & \frac{\partial}{\partial x} & 3x^2\,e^{2y} & \frac{\partial}{\partial y} & \\ & \nearrow & & \searrow & \\ f(x, y) = x^3\,e^{2y} & & & & 6x^2e^{2y} \\ & \searrow & & \nearrow & \\ & \frac{\partial}{\partial y} & 2x^3e^{2y} & \frac{\partial}{\partial x} & \end{array}$$

Linearization and the tangent plane

Photo from pixabay.com

https://pixabay.com/nl/paaseiland-koppen-gezichten-steen-1704991/

The tangent plane



- The tangent plane at $(x_0, y_0, f(x_0, y_0))$ is spanned by the vectors $\mathbf{u} = (1, 0, f_x(x_0, y_0))$ and $\mathbf{v} = (0, 1, f_y(x_0, y_0))$.

The tangent plane

- The tangent plane V at $\mathbf{p} = \big(x_0, y_0, f(x_0, y_0)\big)$ is spanned by the vectors $\mathbf{u} = \big(1, 0, f_x(x_0, y_0)\big)$ and $\mathbf{v} = \big(0, 1, f_y(x_0, y_0)\big)$.

- A normal vector is $\mathbf{n} = \mathbf{u} \times \mathbf{v}$:

$$
\begin{array}{lclllll}
\mathbf{u} & = & (\quad 1, & 0, & f_x(x_0, y_0) \ ) & 1 & 0 \\
\mathbf{v} & = & (\quad 0, & 1, & f_y(x_0, y_0) \ ) & 0 & 1 \\
\mathbf{n} & = & (-f_x(x_0, y_0), & -f_y(x_0, y_0), & 1 \ ) & &
\end{array}
$$

- A normal equation for V is $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = \mathbf{0}$, or

$$z - f(x_0, y_0) = f_x(x_0, y_0) \cdot (x - x_0) + f_y(x_0, y_0) \cdot (y - y_0)$$

The tangent plane

Example

Find an equation for the tangent plane to the surface $x^2 + y^2 + z = 9$ at $P(1, 2, 4)$.

Linearization

- Another equation for the tangent plane is

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \cdot \big(x - x_0\big) + \frac{\partial f}{\partial y}(x_0, y_0) \cdot \big(y - y_0\big)$$

- The tangent plane is the graph of the linear function $L\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$L(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \cdot \big(x - x_0\big) + \frac{\partial f}{\partial y}(x_0, y_0) \cdot \big(y - y_0\big).$$

- This function is called the **linearization of** f **at** (x_0, y_0).

- The function L is an approximation of f in the neighborhood of (x_0, y_0): the closer (x, y) is to (x_0, y_0), the better $L(x, y)$ approximates $f(x, y)$.

Example

With $f(x, y) = 9 - x^2 - y^2$, approximate $f(0.9, 2.1)$ using its linearization at $(1, 2)$. How large is the error?

- Use that the tangent plane to the surface $x^2 + y^2 + z = 9$ is described by the equation $2x + 4y + z = 14$.